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A COURSE OF
EXPERIMENTAL MECHANICS

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EXPERIMENTAL MECHANICS

BY

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WITH AN INTRODUCTION BY

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PREFACE.

PRACTICAL work in Mechanics, besides calling attention to the fact that the subject rests upon an experimental foundation, fulfils an important function in assisting the imagination to grasp the abstract principles of the science; and a few suggestions are here offered as to methods which may be adopted in conducting practical work in this subject with a class of, say, twenty or more students.

When the problem to be considered is sufficiently simple to allow of each student—or even each pair—being provided with a separate set of apparatus, the class may be assembled at the outset of the practical period for a combined discussion of this single problem. The students may be encouraged to say how they would themselves attack the problem, the difficulties which attend it may be discussed; and finally, when alternatives have been considered, the teacher may recommend the method which his own experience has led him to adopt.

In many laboratories, however, it will happen that the number of sets of apparatus available for the elucidation of a given problem is in inverse proportion to its complexity. Thus, at a more advanced stage, there may be many different experiments proceeding at once; and so the method must be modified. The preliminary discussion must take place in a period previous to that in which the practical work is done—hence the need for the student's memory to be refreshed with a written account of the teacher's recommendations. This book has been expanded from a collection of such notes given to the Science Side at Shrewsbury School; and a large number of the practical results were obtained by students themselves.

Some of these experiments may seem, perhaps, to be suitable, in the form in which they are presented, only to the laboratory for which they were devised; in places where the equipment is different, however, the necessary modifications will naturally suggest themselves—and much of the apparatus may be "home-made". An organised system of units, which can be rapidly assembled is, however, a convenience in providing for the needs of a large class with economy of effort; and for this reason the "Capstan" system of units of mechanical apparatus made by Messrs. G. Cussons Ltd., of Manchester, was largely employed in the experiments described in this book.

A detailed list of the apparatus recommended has been placed at the head of each experiment, for the convenience of the laboratory attendant.

It may, perhaps, be argued that refined experiments in illustration of advanced mechanical principles are unnecessary; that students of ability will apprehend these principles with only the rudest practical aid, while those of lesser calibre cannot expect to advance beyond the simplest stage. Many teachers are agreed, however, that even the latter class of student can advance a long way in the understanding of theoretical principles, if they are brought within the range of his sense impressions by appeal to careful experiment. For every kind of student, too, the careful experiment with accurate apparatus is an invaluable means of impressing upon his memory the abstract principle which it illustrates, even if he has little manipulative skill; in doing it he has, as the French word suggests, passed through an experience.

As this course of practical work was designed to accompany a two-years' course of theoretical Mechanics, a suggestion as to which experiments should be left till the second year might perhaps be of use to teachers, and to students working by themselves. These more advanced experiments are indicated in the text by an asterisk, thus:—* EXPERIMENT XXVII.

The experiments are graduated in difficulty from the simplest stage up to University Scholarship standard. They cover the syllabus in Mechanics for the Lower, School, and Higher Certificates of the Oxford and Cambridge Examinations Board, the Oxford and Cambridge Locals Examinations, the London Matriculation, and the Northern Universities Matriculation Examinations; while the whole course, it is hoped, would be a useful preparation for such examinations as the Intermediate and Final B.Sc. at London University. The course also covers the range required in Mechanics for the Army Entrance Examinations at Woolwich and Sandhurst. Again, it is hoped that the book may be useful to Technical students taking Major Engineering Courses.

While designed principally for students working under a teacher, it is also hoped—as already mentioned—that the book may be of use to those who are working alone; and, with this object in view, a collection of definitions and a summary of main principles have been added in Appendix I., while Appendix II. contains some Mathematical notes on the more advanced work. To the student working alone the suggestion is made that he should, before carrying out the method here adopted, consider all the possible ways in which the problem could be attacked; and that he should also consult as many standard works on the subject as are available.

I take this opportunity of expressing my cordial thanks to those of my colleagues who have assisted me: to Mr. C. J. Baker, Senior Science Master, for many valuable suggestions; to Mr. C. W. Stokes for his kindness in reading through all the proofs, verifying the numerical exercises, and suggesting a number of important improvements; and to Mr. P. W. Pilcher for his excellent photographs, from which many of the illustrations have been made.

I am also indebted to Mr. A. E. White, Headmaster of the Borough Technical School, Shrewsbury, for some valuable criticisms.

H. J. E. BAILEY.

SHREWSBURY,
October, 1923.

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(The harder Experiments marked with an asterisk may be left till the second year of the student's course.)

INTRODUCTION.

THERE was once a lecturer who improved his course on theoretical mechanics with experimental demonstrations. One day in his absence someone else kindly offered to take his place and committed himself to showing that the weight W in a machine of pulleys remained steady although other parts moved—all according to theoretical plan. To the delight of the class the demonstration failed, for nothing would induce W to remain still. Whereupon the original lecturer was asked how he had always succeeded when he manipulated the strings; to which he replied '*I nailed W*'!

This is quite typical of the state of mind of mathematical students not so many years ago when they approached Applied Mathematics. There was more than a suspicion that the formulæ would not work in practice, or that the machines and devices which they studied were not of a practical pattern. It was quite the exception for the student to handle models for himself and to find out how far he could approximate to theoretical results, and why and to what extent failure arose. For it was an easy thing to accept the verdict of experts, shaped into convenient Laws of Motion, and to follow the amazingly varied paths which led from these fundamental axioms into the enchanting realm of balance and movement. Also, an appearance of realism could be added by drawing upon the world of living things for subjects in motion. Thus Walton, who quotes Bacon on his title page to *Theoretical Mechanics* (2nd edition, 1885)—' Examples give a quicker impression than arguments'—within fifteen examples in Chapter XIII. invokes the aid of five monkeys, four beetles, two flies, a dog, a mouse, a spider, a snail and other smaller moving creatures.

Nor is such a form of instruction without avail, especially when a certain rough and ready first hand acquaintance with the laws of impulse and momentum is to be had by the whole youth of our land, which has given itself over, heart and soul, to the more violent forms of road transport.

But for all this to produce a proper grasp of mechanics requires from the student considerable intellectual effort at the outset—far more than he is generally willing to give. Consequently the method is of little use to the large and growing number of young students for whom a general knowledge in mechanics is part of their training; it really only serves for the acknowledged mathematician.

The present book, which seeks to illustrate the fundamental assumptions of mechanics by providing a practical series of experiments which the student can try for himself, is therefore of considerable interest. There have been and doubtless there will be other books with much the same purpose; and the danger to which they lay themselves open is the converse of that danger which has already been pointed out; namely, the practical book may be a very imperfect guide to the heart of the subject. A boy may be involved in details of machinery and never acquire a notion of the significance of energy, kinetic or potential.

The author of this *Course of Experimental Mechanics* has steered a judicious course between these two dangers, for he has combined his requisite theoretical grasp with a practical technique. The experiments have been evolved in teaching both the specialising and the ordinary boy at Shrewsbury School, so that what has been described hereafter has borne the test of practice. It is in the belief—to my own mind a well justified belief—that this experience will prove useful in other schools and colleges, that the Author has ventured to put his work into this more permanent form. Certainly in the case of boys proceeding to a college course including Physics or Applied Mathematics, whether at an older or a newer University, it would be an excellent preparation to have embarked on a substantial part of this series of experiments. It leads naturally to more advanced work.

The book will, I consider, also prove interesting and suggestive in another part of the educational world. In Technical Institutions and among the more advanced students in the Evening Classes, and in Junior Technical Schools there is a considerable need for books such as this, which link up the elementary practical work in fitting stages with the general underlying principles. Very great ingenuity is often displayed both by teacher and by pupil in devising apparatus, frequently home-made, but the ingenuity usually needs purposeful guidance. The scheme of this book is elastic enough to meet this need without discouraging initiative and originality.

In a Lancashire Day Continuation School—an early product of the 1918 Education Act—an Irish boy was asked to name his favourite subject. 'Mathematics,' he replied, 'because I haven't got to learn it.' Further enquiry revealed that he could *reason* it all out and not learn it by heart; and this was what appealed to him. Now his mathematics was a course very like the earlier parts of this book, although his previous elementary schooling had been very much interrupted. This certainly suggests that the methods here used are very fruitful even on what seems unfavourable ground.

Incidentally, the book has several interesting features—the hints concerning practical experimental difficulties, the historical notes, and the summaries of mathematical formulæ and physical principles.

I think it is a distinctly useful addition to the stock of elementary textbooks; and one which will supply a general want.

H. W. TURNBULL.

St. Andrews,
October, 1923.

EXPERIMENT I.

LAW OF THE LEVER.—CASE I.

Object of Experiment:

To show that two forces F_1, F_2, acting at right angles to a uniform rod AB balanced horizontally at its centre O, will maintain the rod in horizontal equilibrium, if their distances x, y from O are such that

$$F_1 x = F_2 y.$$

Apparatus:

Drilled metre rule, Capstan stand and block, pivot, slotted weights and 2 hangers, thread. small retort-stand, 2 metal rods, small piece of copper wire.

Method:

(1) Attach the pivot to the Capstan block, and hang the rule on the pivot by the centre hole O. (If the rule does not balance horizontally, bring it to the horizontal position by hanging a small piece of copper wire on the rule, and sliding it about till the rule is brought to the horizontal position. In Fig. 1 this is called the counterpoise K.)

Place the small retort-stand with two rods projecting from it—as shown in Fig. 1—just above and below one end of the rule, to prevent it swinging right up or down.

(2) Make two loops of thread about 2 in. long to slip over the rule, one on each side of O.

(3) **F_1 and x fixed: F_2 and y variable:**

(*a*) Put X at the 5 cm. scale-reading of the rule, and attach 0·25 lb. wt. to it.

Attach to Y successively 0·25, 0·3, 0·4, . . . 1·0 lb. wt., in each case adjusting the position of Y for equilibrium.

In each case note the distances x and y.

Tabulate your results as in Table 1.

(*b*) Put X at the 10 cm. scale-reading, and repeat the above operations.

Tabulate your results; *working out the calculations at the time.*

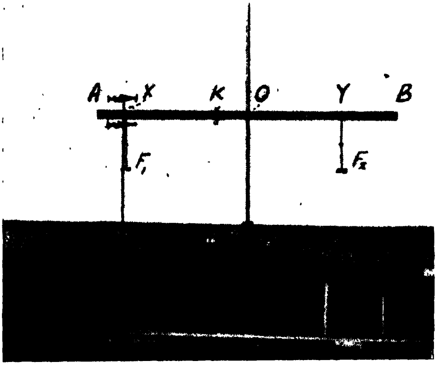

FIG. 1.—OX = x. OY = y.

(4) **x and y fixed; F_1 and F_2 variable:**

Place X and Y at any two points of the rule (say $x = 35$ cm., $y = 45$ cm.).

Take F_1 any weight, and obtain a balance by adjusting the size of F_2 by steps of 0·01 lb., X and Y remaining fixed.

In this way take five observations, X and Y remaining the same throughout.

Table I. below, which gives the results of a few actual experiments selected at random, will serve as a guide in the tabulation of readings.

TABLE I.

Experimental Results.

Number of Experiment	1	2	3	4, etc.
F_1 (lb. wt.)	0·25	0·25	0·25	
x (cm.)	45·00	45·00	45·00	
F_2 (lb. wt.)	0·25	0·50	0·70	
y (cm.)	45·00	22·60	16·00	
F_1x	11·25	11·25	11·25	
F_2y	11·25	11·30	11·20	
and its % diff. from F_1x	0·00	+0·11	−0·11	

Experimental Problems on Experiment I.

1. Find the pressure on the pivot, by tying a loop of thread through O; and suspending the rule from it by a spring-balance, in each of the above cases.
   Is the pressure upward or downward?
2. Find the weight of the rule by balancing it about a point other than O.
3. Find the relation between gms. and lb. by hanging gms. on one side and lb. on the other; the rule being pivoted at O.

EXPERIMENT II.

LAW OF THE LEVER.—CASE II.

Object of Experiment:

This case differs from Case I. in that F_2 acts obliquely to AB. It is the object of the experiment to show that the condition for horizontal equilibrium of AB is

$$F_1x = F_2q,$$

where q is the length of the perpendicular from O on the line of action of F_2.

Apparatus:

Drilled metre rule, Capstan stand, 2 Capstan blocks, pulley, pivot, 2 hangers and slotted weights, thread, small retort-stand, 2 metal rods, set-square, short ruler, small piece of copper wire to counterpoise the rule.

Method:

(1) Fit up the apparatus as in fig. 2*a*. If the unloaded rule does not balance horizontally, counterpoise it, as in Experiment I., Method (1).

The pulley should be placed about half-way between O and the base of the stand. Care should be taken that the groove of the pulley is accurately in the plane of the rule; and also that the guide of the pulley does not scrape against the oblique thread.

F_1 is hung from AO by a loop of thread, as in Experiment I.

F_2 is made to act obliquely on the rule by the thread F_2Y.

(2) Place X at the 5 cm. scale-reading, and attach 0·25 lb. wt. to it. Take Y at the 80 cm. scale-reading.

(3) Adjust F_2 by steps of 0·01 lb. until AB balances horizontally.

(4) Before measuring the length of the perpendicular q, clamp the rule in the equilibrium position by means of the two rods projecting from the small retort stand.

The length of q may be found with a set-square and ruler; but the following method is much quicker, and at least as accurate :—

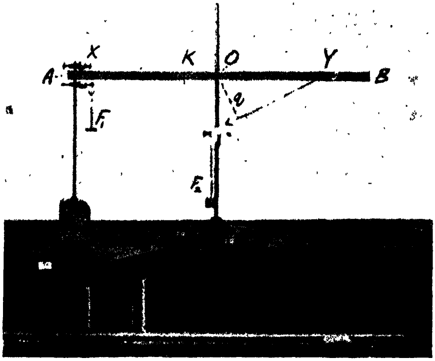

Fig. 2*a*.

Make a loop on a piece of thread, and hang it on the pivot. Now grip the thread between the finger-nails of the thumb and first finger, and find by trial the length of thread such that, when the latter is swung about the pivot as centre, the point of the thread which is held between the finger-nails just grazes the string F_2Y. (Position I., fig. 2*b*.) This is the length of the perpendicular q.

To measure its length, swing the thread up until it lies along the rule (Position II., fig. 2*b*), and read the length off against the scale.

(5) Perform similar experiments with X at the 10, 15, 20 . . . 45 cm. scale-readings. Tabulate your results, *working out the calculations at the time.*

(6) Put Y at the 90 cm. scale-reading; change F_1 to 0·3 lb. wt., and repeat operations (2) to (5).

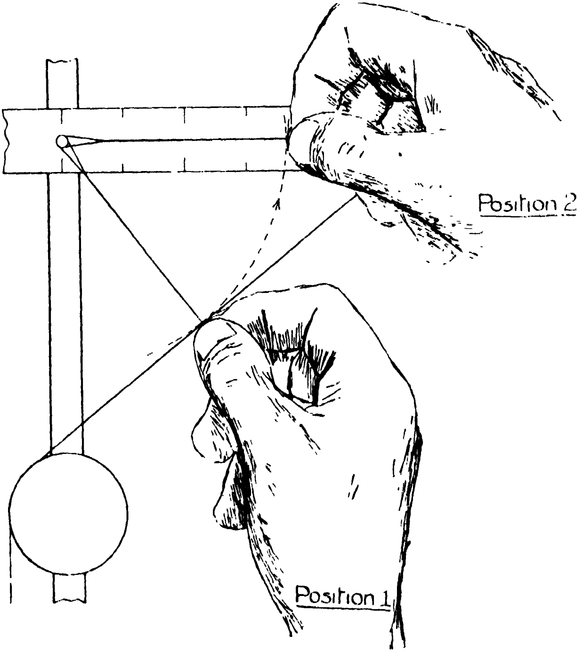

Fig. 2*b*.

(7) Repeat the above operations with F_1 and y any values you like.

Table II. below, which gives the results of a few actual experiments selected at random, will serve as a guide in the tabulation of readings.

TABLE II.

Experimental Results.

Number of Experiment	1	2	3	4, etc.
F_1 (lb. wt.)	0·25	0·45	0·40	
x (cm.)	40·0	40·0	40·0	
F_2 (lb. wt.)	0·46	0·82	1·06	
y (cm.)	35·0	35·0	30·0	
q (cm.)	21·9	21·9	15·0	
F_1x	10·00	18·00	16·00	
F_2y	16·10	28·70	31·80	
F_2q	10·07	17·96	15·90	
and its % diff. from F_1x	+0·70	−0·22	−0·63	

EXPERIMENT III.

LAW OF THE LEVER.—CASE III.

Object of Experiment:

In this case both forces act obliquely to AB. It is the object of the experiment to show that the condition for horizontal equilibrium of AB is

$$F_1p = F_2q,$$

where p and q are the lengths of the perpendicular from O on the lines of action of F_1 and F_2 respectively.

Apparatus:

Drilled metre rule, Capstan stand, pivot, 3 *Capstan blocks,* 2 *pulleys,* 2 *hangers and slotted weights, thread, small retort-stand,* 2 *iron rods, set-square, short ruler, small piece of copper wire to counterpoise the rule.*

Method:

(1) Fit up the apparatus as in fig. 3*a*.

If the unloaded rule does not balance horizontally, counterpoise it, as in Experiment I.

F_1 and F_2 are made to act obliquely on AB by the threads XT_1, YT_2 passing over two pulleys.

These pulleys may be in any convenient position—say 10 and 20 cm. from O respectively.

(2) Place X at the 15 cm. scale-reading, and attach 0·25 lb. wt. to it.

(3) Place Y at the 80 cm. scale-reading, and adjust F_2 for horizontal equilibrium of AB.

(4) Using the method explained on p. 5, measure p and q, the perpendiculars from O on XT_1 and YT_2 respectively.*
Tabulate your results, as in Table III.

(5) Repeat operations (2) to (4) with F_1 successively 0·4, 0·6, 0·8 lb. wt.

(6) Alter the positions of P_1 and P_2 (without changing X and Y), and repeat operations (2) to (5).

(7) Repeat operations (2) to (6) with X and Y at different places.

All readings should be tabulated, and the calculations worked out at the time.

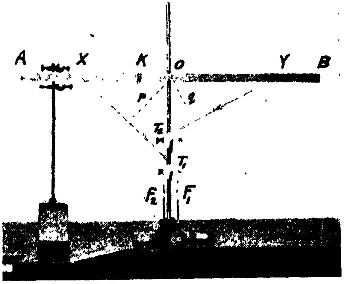

Fig. 3*a*.

Table III., p. 10, which gives the results of a few actual experiments selected at random, will serve as a guide in the tabulation of readings.

Experimental Exercise.

Set up apparatus as in fig. 3*b*, using any weights you please which will produce horizontal equilibrium of the rule.

Calculate the pressure on the pivot, and then verify that the sums of the clockwise, and of the anticlockwise, moments of all the forces about any point in their plane are equal.

---

* T_1, T_2 are the points of contact of the left and right strings with their pulleys.

TABLE III.

Experimental Results.

Number of Experiment	1	2	3, etc.
F_1 (lb. wt.)	0·4	0·6	
x (cm.)			
p (cm.)	12·4	11·6	
F_2 (lbs. wt.)	0·25	0·35	
y (cm.)			
q (cm.)	20·0	20·0	
F_1x			
F_1p	4·96	6·96	
F_2y			
F_2q	5·00	7·00	
and its °/. diff. from F_1p	+0·81	+0·57	

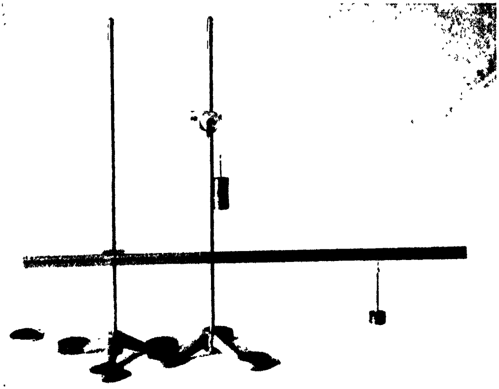

Fig. 3*b*.

EXPERIMENT IV.

BORDA'S COUNTERPOISE METHOD OF WEIGHING.

This method of weighing was introduced by Jean Charles Borda (1733-99). It eliminates the error due to the inequality of the arms of a balance. It should therefore always be employed when great accuracy is desired.

Apparatus:

Capstan stand and block, pivot, drilled metre rule, 2 *scale-pans, string, slotted weights, unknown weight, counterpoise obviously greater than the unknown weight, small retort-stand and* 2 *iron rods,* 2 *small hooks.*

Method :

(1) Make two loops of string about 2 in. long to slip over the beam AB, and hang scale-pans from these by two hooks.

(2) Place X and Y near the ends of AB, but not equidistant from the pivot.

(3) Place in the scale pan L (fig. 4) the unknown weight Z whose value is desired.

(4) Place in the scale-pan R a counterpoise—whose weight need not be known—obviously heavier than Z.

(5) Add weight W_1 to L so that AB balances horizontally.

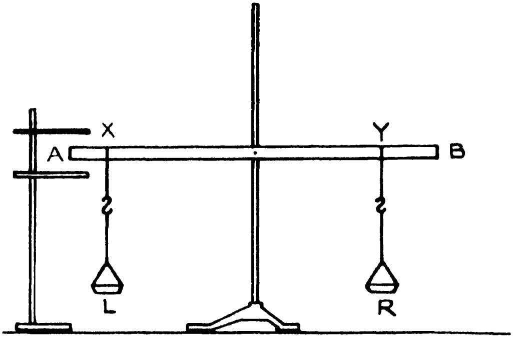

FIG. 4.

(6) Remove Z from L, and substitute weight W_2, so that AB is again horizontal.

Then $W_1 + Z = W_2$.

or $Z = W_2 - W_1$,

a result not depending on the length of the arms XO, YO.

(7) Check the accuracy of the above value of Z by repeating operations (2) to (6) with X and Y at different places.

EXPERIMENT V.

EQUILIBRIUM OF A LOADED BEAM.

The beam is suspended horizontally from two spring-balances, and weights are hung on it at intervals.

It is the object of the experiment to show that the algebraic sum of the moments about any point of all the acting forces is zero.

Apparatus:

Wooden beam fitted with two hooks and placed across two stools, drilled metre rule, two spring-balances, two ring weights, thin string.*

Method:

(1) Fit up the apparatus as in fig. 5*a*. Take C, X, Y, D at the 10, 40, 75, 85 cm. scale-readings respectively.

(2) Before attaching the ring weights, note the readings of the spring-balances. The sum of these gives W lb., the weight of the rule. Take W as acting at O.†

(3) Attach (say) 2 lb. and 4 lb. ring weights to X and Y, and note S_1 and S_2, the readings of the spring-balances.

(4) **Calculation of the moments:**

Taking moments about A, we have

Clockwise moments produced by the forces F_1, W and F_2:

Anticlockwise moments by the forces S_1 and S_2.

Taking moments about C, we have

Clockwise moments produced by the forces F_1, W and F_2:

Anticlockwise moments by the force S_2.

As the force S_1 passes through C, it has no moment about C.

The following details from an actual experiment will show the most convenient method of tabulating results :—

---

* Or, alternatively, as in fig. 5*a*.

† O is the point of balance of the unloaded beam, which may be determined experimentally beforehand. In this case it was the mid-point of the beam.

Observed values of the acting forces.

$S_1 = 2{\cdot}85$ lbs. wt.
$S_2 = 3{\cdot}55$,, $S_1 + S_2 = 6{\cdot}40$
$W = 0{\cdot}4$,,
$F_1 = 4{\cdot}0$,,
$F_2 = 2{\cdot}0$,, $F_1 + F_2 + W = 6{\cdot}40$

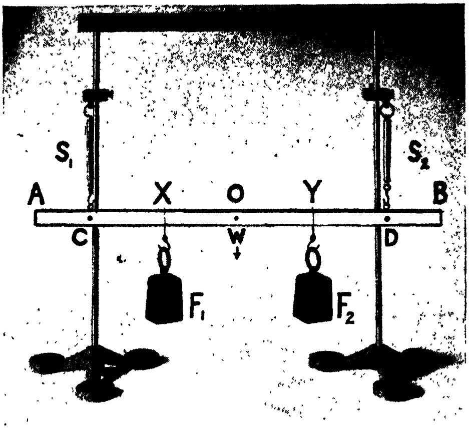

FIG. 5a.

Moments about A :

(i) **Clockwise**

$AX.F_1 = 40 \times 4 = 160$
$AO.W = 50 \times 0{\cdot}4 = 20$
$AY.F_2 = 75 \times 2 = 150$

Total 330

(ii) **Anticlockwise**

$AC.S_1 = 10 \times 2{\cdot}85 = 28{\cdot}5$
$AD.S_2 = 85 \times 3{\cdot}55 = 301{\cdot}75$

Total $330{\cdot}25$

°/₀ diff. between (i) and (ii) = 0·08

Moments about C :

(i) **Clockwise**

$CX.F_1 = 30 \times 4 = 120$
$CO.W = 40 \times 0{\cdot}4 = 16$
$CY.F_2 = 65 \times 2 = 130$

Total 266

(ii) **Anticlockwise**

$CD.S_2 = 75 \times 3{\cdot}55 = 266{\cdot}25$

°/₀ diff. between (i) and (ii) = 0·09

The total moments about the points X, O, Y, D, and B should also be calculated by the student in a similar manner.

Problem:

What influence upon the accuracy of the results will be introduced by having the rule slightly out of the horizontal?

Experimental Exercise.

You are provided with an irregularly-shaped flat board fitted with a number of projecting pins (fig. *5b*). The board is pivoted about its centre of gravity, so that it will remain in any position in which it is set.

Set up the apparatus as indicated in the figure, attaching weights to each of the pins.

To the pin which is in line with the pivot, attach a plumb-line.

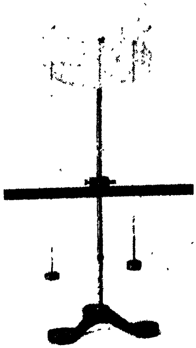

FIG. *5b*.

Using the horizontal scale to measure the arms of the moments, verify that the sums of the clockwise and anticlockwise moments of the weights about the pivot are equal.

The experiment should be repeated, using different weights.

EXPERIMENT VI.

THE STEELYARD (ROMAN).

Object of Experiment:

(*a*) **To test the accuracy of the graduations on each side;**
(*b*) **To find the relation between kilogrammes and pounds;**
(*c*) **To find the position of the centroid of the Steelyard.**

Apparatus:

*Steelyard, large beam fitted with strong hook and supported on stools—or otherwise**—*so as to be about 4 ft. from the ground,* 1, 2, 4, 7, 14 *and* 28 *lbs. weights, slotted kilogramme weights and hanger, kilogramme spring-balance, short rule.*

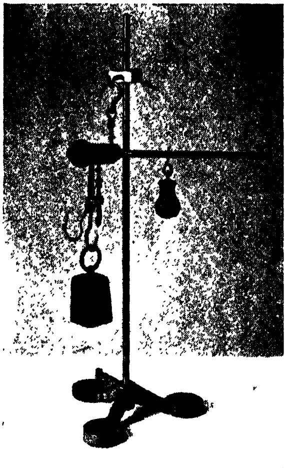

FIG. 6*a*.

Explanation of the Instrument:

Fig. 6*b* is a diagram of the essential features of the Roman steelyard, of which fig. 6*a* is a photograph.

Here H is the pivot from which the load is suspended by a hook *h* (shown on the left in fig. 6*a*). N_1, N_2 are two knife-edges, on either of which the instrument may be balanced; N_1 being for smaller, and N_2 for larger weights.

k denotes the weight of the counterpoise and K the point on the arm of the steelyard at which it acts.

---

* The particular steelyard described here was graduated from 0-18 lbs. wt. on one edge of the arm, and from 18-60 lbs. wt. on the reverse edge. The one shown in fig. 6*a* is rather smaller than this.

z denotes the effective weight of the steelyard and its attachments (excluding the counterpoise), when the knife-edge N_2 is used.

Method:

(*a*) (1) **To test the accuracy of the graduations for light weights:**

Hang the steelyard from the hook in the beam so that the knife-edge N_1 is employed. In this case the graduations 0 — 18 lbs. wt. are uppermost.

Attach successively 2, 3, 4, etc., lbs. wt. to *h* (fig. 6*a*), and adjust *k* for horizontal equilibrium, noting the scale-reading of K in each case.

Tabulate your results as in Table VI*a*.

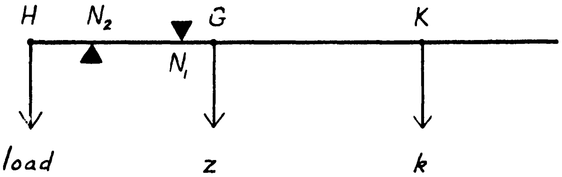

FIG. 6*b*.

(2) **To test the accuracy of the graduations for heavier weights:**

Reverse the steelyard, so that the knife-edge N_2 is employed. In this case the graduations 18 — 60 lbs. are uppermost.

Attach successively 18, 19, 20, etc., lbs. wt. to *h*, and proceed as in (1).

Tabulate your results, as in Table VI*b*.

(*b*) **Find the relation between kilogrammes and pounds as follows:**

(1) Using N_2, hang 9 kilogrammes on the hook, and adjust *k* for horizontal equilibrium, noting the scale-reading of K in each case.

From this calculate the number of lbs. in 1 kg.

(2) Repeat the operation, increasing the load by 1 kg. at a time up to the limit of the hanger.

Tabulate your results, as in Table VI*c*.

(*c*) **Find the position of the centroid G of the steelyard as follows:**

(1) Weigh the counterpoise k alone with a spring-balance.

To do this it is not necessary to detach the counterpoise. Instead, lift it up with the hook of the spring-balance until it is clear of the arm of the steelyard.

(2) Similarly weigh the hook by which the steelyard is suspended.

(3) Weigh the whole steelyard and counterpoise.

Hence obtain the weight z kilogrammes of the steelyard alone (excluding the hook by which it is suspended).

(4) Balance the steelyard about N_2 with a load of 10 kilogrammes. Measure the distances HN_2, KN_2 with a rule.

The position of G may then be calculated by the principle of moments as follows:—

Taking moments about N_2 (fig. 6*b*), we have

Clockwise moments: $N_2K \cdot k = \ldots$ (units of moment)
$N_2G \cdot z = \ldots$ (,,)

Total clockwise moments $= \ldots$ (,,)
Anticlockwise moments: $10 \cdot HN_2 = \ldots$ (,,)

The only unknown quantity in the above is N_2G—the distance of G from the knife-edge N_2—which is obtained by equating clockwise and anticlockwise moments.

(5) Check the accuracy of the above value of N_2G by repeating the experiment with a load of 15 kilogrammes.

Experimental Results.

TABLE VIa.—Test of graduations for lighter weights.

Observation No.	1	2	3, etc.
Load (lbs.)	2	3	4, etc.
Scale reading			
Error %			

TABLE VIb.—Test of graduations for heavier weights.

Observation No.	1	2	3, etc.
Load (lbs.)	18	19	20, etc.
Scale reading			
Error %			

TABLE VIc.—Relation between kilogrammes and pounds.

Observation No.	1	2	3, etc., up to 12
Kilogrammes	9	10	11, etc., up to 20
Equiv. in lbs.			
Value of 1 kg. in lbs. ...			

Mean value of 1 kilogramme = ... pounds.

The student would also find it instructive to represent on a graph the results recorded in each of his Tables VI*a* to VI*c*.

EXPERIMENT VII.

THREE FORCES IN EQUILIBRIUM MEET IN A POINT.

Object of Experiment:

(*a*) **To show experimentally that the lines of action of 3 forces P, Q, R, which act in one plane upon a body so as to keep it in equilibrium, meet in a point O; and that**

(*b*) **Each force is proportional to the sine of the angle between the other two, i.e. that**

$P/\sin QOR. = Q/\sin ROP. = R/\sin POQ.$[*] [†]

(see fig. 7*a*).

Apparatus:

Flat glass plate in a wooden frame fitted with levelling-screws, thick glass plate of irregular shape fitted with 3 *small brass clamps,* 3 *steel spheres,* 3 *retort-stands,* 3 *retort-stand pattern pulleys,* 3 *hangers and slotted weights, spirit-level, small table* (*or* 4 *piles of wood blocks each* 1 *ft. high*).

Method:

(1) **Levelling of glass frame:** Place the frame on the small table (or on wood blocks) so as to be about 1 ft. above the bench.

---

[*] This result is known as *Lami's Theorem.* For the purposes of calculation the equivalent

$$P \,.\, \mathrm{cosec}\ QOR = Q \,.\, \mathrm{cosec}\ ROP = R \,.\, \mathrm{cosec}\ POQ,$$

is often more convenient.

[†] The oblique sign / is freely used throughout the book, where convenient, to separate the numerator and denominator of a fraction. In cases where there might be ambiguity, the end of the denominator is followed by a full stop.

Put the spirit level in the centre of the frame along AC (fig. 7*b*), and adjust the screws A and C until the bubble is central.

Now set the spirit-level along BD, and adjust these screws until the bubble is again central.

The glass plate should now be level.

Next place the 3 steel spheres on the glass near the centre; if they do not remain where they are placed, make a final adjustment of the levelling-screws until they do.

Each screw should now be clamped by means of the nut underneath it.

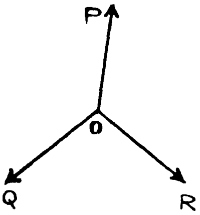

FIG. 7*a*.

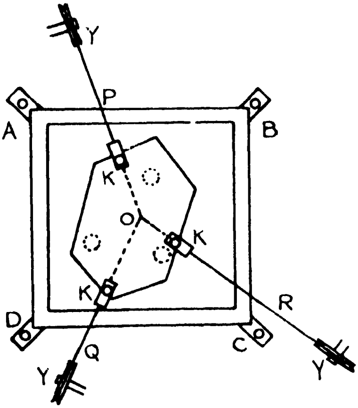

FIG. 7*b*.

(2) Cut a piece of paper the shape of the irregular plate, and place it on top.

Each of the three clamps K should be provided with a thread by which to attach it to the hanger, passing over a pulley, Y.

In addition, a piece of thread about 8 inches long should be attached to the opposite end of each clamp through the hole provided, the other end of the thread remaining loose, as in fig. 7*c*.

(3) Attach a hanger to each clamp by a thread passing over a pulley; the clamp thus serves the additional purpose of gripping the paper.

Each hanger should be supported on small wood blocks until a suitable balance is obtained.

(4) Take P and Q any convenient magnitudes. (In the experiment detailed below $P = 0{\cdot}28$, $Q = 0{\cdot}30$ lb. wt.).

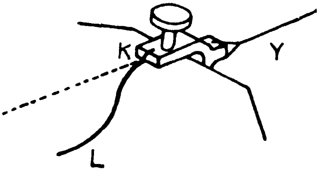

FIG. 7c.

Increase R in steps of 0·01 lb. until the other two hangers are just lifted off their blocks.

Readjust the stands so that the plane of each pulley is parallel to its string.

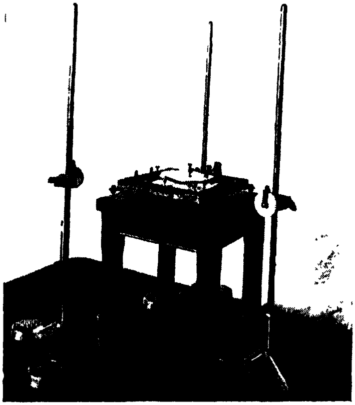

FIG. 7d.

Fig. 7d shows the apparatus adjusted for taking an observation.

(5) **Plotting directions of P, Q and R:** Without disturbing the balance, carefully set each thread KL (fig. 7c) in a straight line with KY.

With a fine pencil mark two points on the paper vertically under KL.

(6) Remove the paper and draw lines through the three pairs of points marked. These three lines should be concurrent, as in fig. 7*a*.

(7) Measure carefully the angles *QOR*, *ROP* and *POQ*, with a protractor. It should be possible to estimate each angle to a quarter of a degree.

(8) Repeat the experiment with different weights on the hangers.

The following details of an experiment show a convenient method of tabulating results:—

TABLE VII.

Experimental Results.

No. of Experiment.	1			2, etc.
P (lbs. wt.)	0·28			
Q („)	0·30			
R („)	0·44			
Angle *QOR*	141° = 180° − 39°			
„ *ROP*	138° = 180° − 42°			
„ *POQ*	81°			
Cosec *QOR* *	1·59			
cosec *ROP*	1·50			
cosec *POQ*	1·01	Diff.	Error °/₀	
P cosec *QOR*	0·45	+ 0·003	+ 0·7	
Q cosec *ROP*	0·45	+ 0·003	+ 0·7	
R cosec *POQ*	0·44	− 0·007	− 1·6	
Mean of the above 3 from which the differences are taken	0·447			

---

* The student should, in all practical work, guard against the temptation to claim a greater degree of accuracy than his apparatus warrants. It must here be remembered that the weights can only be varied in steps of ·01 lb., while the angles are only claimed to be accurate to ¼°. The values of the cosecants of the angles cannot therefore be relied on to more than two places of decimals.

EXPERIMENT VIII.

THE TRIANGLE OF FORCES.

Object of Experiment :

To show that if three forces act in one plane upon a body so as to maintain it in equilibrium, they may be represented, in magnitude and direction, by the sides of a triangle, taken in order.

Apparatus :

Wooden beam about 5 feet long fitted with 2 hooks and supported on two stools (alternatively—2 Capstan stands and blocks, 2 hooks for these, wooden tie-beam, 4 kilogramme weights), 1 Capstan stand and block, wooden board, 3 pieces of string knotted together, 2 spring-balances reading to 4 lbs. wt., set-square, ruler, drawing-pins, 4-lb. ring weight.

Method :

(1) Tie the 4-lb. weight to one end of the 3-way string, and to each of the other ends tie a spring-balance.

In order that the spring-balance readings may not be equal, the two upper strings should be of unequal lengths.

(2) Hang the spring-balances from the hooks in the wooden beam (or, alternatively, as in fig. 8*a*).

(3) Attach a piece of paper to the board, and screw the board to the block on the Capstan stand.

Adjust the stand so that it is just behind the string, the knot being a little below the centre of the paper.

(4) With the set-square and a sharp pencil mark two points perpendicularly behind each of the three strings.

Note the spring-balance readings S_1 and S_2.

(5) Unscrew the board from the stand. Join the three pairs of points with straight lines. If the points have been recorded accurately the three lines so obtained should meet in a point O (fig. 8*b*).

(6) **Derivation of the triangle of Forces:** Taking 4 cm. to represent 1 lb., produce WO (fig. 8*b*) backwards to P so that OP represents W to scale.

From P draw PQ parallel to S_1O to meet OS_2 in Q.

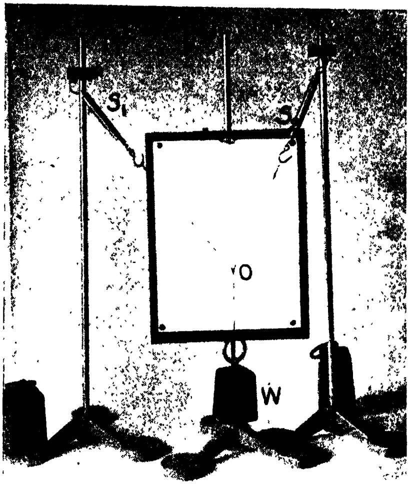

FIG. 8*a*.

Measure the sides PQ, OQ of the triangle OPQ. Each of these sides should be numerically 4 times the value of the corresponding force.

The results of the experiment should be recorded as in Table VIII. below. In this table the percentage errors are estimated on the *calculated* values of the sides.

(7) Repeat the experiment with different values for the forces. This may be secured by altering the load, the lengths of the strings, or the distance between the hooks.

TABLE VIII.

Experimental Results.

Experiment No.	Forces (lbs. wt.)	Corresponding sides of triangle. Calculated.	Observed.	Error °/₀
1	W 4·0 S_1 2·9 S_2 3·7	16·0 11·6 14·8	16·00 11·75 15·00	— + 1·3 + 1·4
2, etc.				

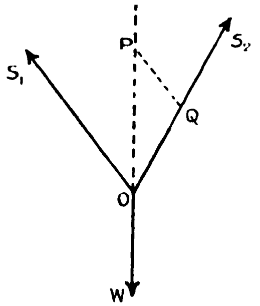

FIG. 8b.

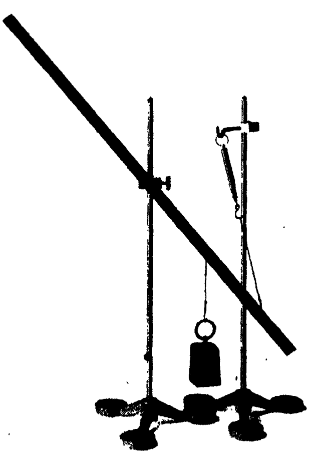

FIG. 8c.

Experimental Exercises.

1. By measuring on your paper the angles between the lines WO, S_1O, S_2O, verify the relation (*b*) of EXPERIMENT VII.

2. Repeat Experiment VIII., using pulleys and slotted weights instead of the two spring-balances. While requiring a little more skill in adjustment, this arrangement of apparatus should give smaller percentage errors than those in Table VIII.

3. Set up apparatus as in fig. 8*c*, attaching the weight and spring-balance to any points you please.

Calculate the magnitude and direction of the reaction on the pivot.

EXPERIMENT IX.

THE JIB-CRANE.

Object of Experiment:

To measure experimentally the magnitudes of the forces called into play in the several members of a jib-crane when supporting a given load, and to compare them with the values calculated from the Principle of the Triangle of Forces.

Apparatus:

Wooden jib fitted with spring-balance reading to 10 lbs. wt., another similar spring-balance, 2 Capstan stands, 3 Capstan blocks, ¼-inch iron rod, wooden tie-beam, 1, 2 and 4 lbs. ring weights, metre rule, plumb-line.

Description of Apparatus:

The crane consists of a wooden beam DCB (fig. 9*a*)—called the jib—fitted at the end with two fixed pins D and B, and also carrying a movable rod C which slides in slots cut in the beam.

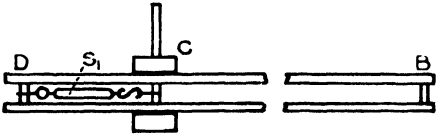

FIG. 9*a*.

The rod C is attached to a block on a Capstan stand. A spring-balance S_1, connecting C and D, records the thrust in the jib when it is supporting a load.

The jib is held up by a string fastened to another spring-balance S_2, attached to a second Capstan block on the same stand (fig. 9*b*). The pin B carries a hook from which the load is suspended.

Method:

(1) Fit up the apparatus as in fig. 9*b*, putting the rod C in the special hole in the lower Capstan block.

Place the iron rod A in the corresponding hole in the upper block, and adjust the latter so that the two rods are parallel, with their centres 30 cm. apart.

(2) With a plumb-line test whether the upper rod is in the same vertical plane as the lower. If not, suitably tilt the stand by putting a wedge of paper under one of the feet until it is.

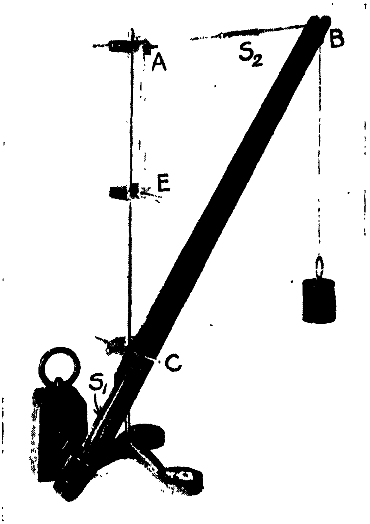

FIG. 9*b*.

(3) Attach the spring-balance S_2 by a string to the pin B. Attach its other end by a second string to the rod A.

It is convenient to have the string AB of such a length that the jib when unloaded makes an angle of about 30° with the vertical.

A simple method of adjusting the length of *AB* is provided by looping the string over the intermediate block E, which carries a projecting pin. The string must, however, be securely fastened at A, so that the tension in *AB* is not communicated to *AE*.

(4) Note the zero readings of the spring balances S_1 and S_2.

(5) Attach a load of 3 lb. to the hook on B. Note the new readings of the spring-balances.

(6) Measure the lengths of AB and BC in centimetres.

All readings should be tabulated at the time.

(7) Repeat operations (4) to (6) with loads of 2 lbs. and 1 lb. successively, the distance AC remaining the same.

(8) Place a load of 4 lb. on the hook, and move A to a fresh position. Care must be taken that this position is such that the spring-balance S_1 is not extended beyond the limit of its scale. The precautions in (1) and (2) must also be observed.

(9) With A in the same position as in (8), take readings for loads of 3, 2, and 1 lb. respectively.

The following details of an experiment show a convenient method of tabulating results :—

TABLE IX.

Experimental Results.

Experiment No.	Value of the forces (in lbs. wt.)			
	Apparent reading,		Zero reading.	Effective reading.
	S_1	6·25	1·750	4·500
	S_2	4·00	0·675	3·375
	Load = 2 lbs.			
	Lengths of the sides.			
1, etc.	Observed.		Calculated.	Error %.
	AC	30·0	30·000	—
	AB	50·8	50·625	+ 0·34
	BC	68·0	67·500	+ 0·74

(The spring-balances used in this experiment were graduated in ounces.)

Problem :

In the above experiment the weight of the jib and its spring-balance was neglected.

Make experiments to determine the weight, and the position of the centre of gravity, of the jib and its spring-balance.

By making allowance for this additional factor, alterations will be produced in the calculated values of the sides AB, AC. What are now the percentage errors of column 3 of Table IX. above ?

EXPERIMENT X.

RESOLVED PARTS* OF A FORCE IN DIRECTIONS INCLINED TO THE LINE OF ACTION OF THE FORCE.

Object of Experiment:

To show that

(*a*) the resolved part of a weight W (lbs. wt.) in a direction inclined at an angle θ to the vertical is

$$W \cdot \cos \theta.$$

(*b*) The resolved part perpendicular to the above direction is

$$W \cdot \sin \theta.$$

Apparatus:

Roller on inclined plane, 3 Capstan stands, 3 Capstan blocks, 2 pulleys, slotted weights and 2 hangers, set-square, thread and lead weight to form plumb-line, spirit level, stand and clamp to hold set-square.

Method:

In order to measure these two resolved parts, the roller is fitted with two stirrups, to the middle points of which are attached two pieces of thread.

The roller is placed on a plane which has been carefully slanted so as to be at the required angle of inclination θ to the vertical.

The two threads pass over the pulleys X, Y, and carry adjustable counterpoising weights at their ends (fig. 10*a*).

By altering the positions of X and Y, as also the size of the counterpoising weights, the apparatus is so adjusted that the roller is held in equilibrium just clear of the plane with the threads WX, WY respectively parallel and perpendicular to the plane.

---

* The term "resolved parts" is usually confined to "components" in mutually perpendicular directions.

The counterpoising weights K and R are then the measures of the resolved parts of W in these two directions; thus it is the object of the experiment to show that

$$R = W.\cos\theta,$$

and

$$K = W.\sin\theta,$$

within the limits of error imposed by the accuracy of the apparatus.

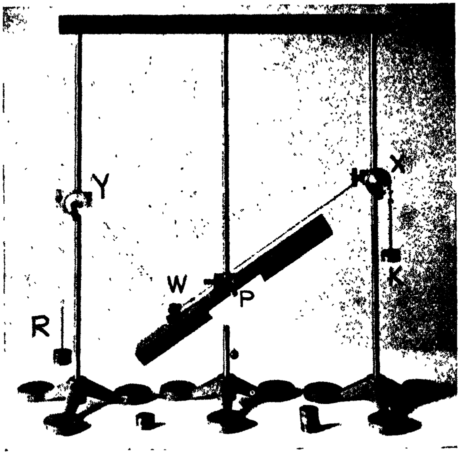

FIG. 10*a*.

Procedure.

(1) Place the roller and its two stirrups on a strong balance (a delicate chemical balance should not be used), and weigh it with the slotted weights correct to 0·01 lb.

(2) Attach the plane to a block on a Capstan stand by means of the special clamping screw.

(3) Suspend a plumb-line formed of the lead weight and a short piece of thread looped at the end from the pin P (fig. 10*a*).

(4) Set the plane at an angle of 20° to the vertical.

> *(Caution: In doing this, take care that the centre of the thread is accurately against the 20° mark on the scale. The thread should hang just clear of the scale; if it is too far forward a parallax error will be introduced.)*

(5) Having clamped the plane firmly at this angle, place the spirit-level transversely on the plane where the roller is to be, to see if the plane is horizontal in this direction.

If it is not, this can be corrected by placing a suitable thickness of paper under one of the feet of the stand. The stand should be set relative to the plane so that a single foot projects at right angles to the length of the plane.

(6) To each of the stirrups attach a thread carrying at its other end a hanger.

Pass these threads over the pulleys X and Y (fig. 10*a*).

Now set X roughly in the proper position, and add weights to the right hand hanger until the roller will remain stationary at any point of the plane at which it is set.

(7) A more careful adjustment must now be made so that the thread WX is accurately parallel to the line of greatest slope of the plane.

This may be done most conveniently by moving the pulley stand about until the thread is correctly slanted. To check the correctness, see if it is possible to stand in such a position that the line of the thread coincides with an edge of the plane.

(8) The position of the pulley Y, and the size of the weights on the left-hand hanger must now be adjusted so that the roller is just lifted off the plane (test whether a piece of paper may be slipped between the roller and the plane) when the angle XWY is accurately a right angle.

In order to get the angle XWY between the strings exactly a right angle, set up a large set-square fixed in a stand in front of the plane. Adjust it so that one edge may be viewed as coincident with the thread WX, and at the same height.

Now make final adjustments of both stands and both weights until the thread WY may be viewed as coincident with the second edge of the set-square which forms its right angle.

(9) Lastly, view the four portions of the thread edgeways, to test whether they all lie in the same vertical plane. Take care that the grooves of the two pulleys are also in this plane.

(10) It will be observed that the inclined plane is merely a device for setting the threads at the required slant, and does not affect the final result. Thus no allowance has to be made for roller friction.

(11) Repeat the experiment as often as time allows, with the plane set at different angles.

Table X below, which contains the details of an actual experiment, shows a convenient method of tabulating results. These latter should be recorded, and the calculations made, *at the time*.

TABLE X.

Experimental Results.

Observation No.	1	2, etc.
θ	20°	
Sin θ	0·34	
W (lbs. wt.)	1·15	
$W \sin \theta$	0·39	
K (lbs. wt.)	0·39	
and its % diff. from $W \sin \theta$	0·00*	
Cos θ	0·94	
$W \cos \theta$	1·08	
R (lbs. wt.)	1·08	
and its % diff. from $W \cos \theta$	0·00*	

* That is, no error within the limits of accuracy imposed by the fact that the smallest weights used are 0·01 lb.

Experimental Exercise.

Set up apparatus as in fig. 10*b*, with the string which supports the roller horizontal. (The string should be passed through the hole in the plane, care being taken that it is quite clear of the sides of the hole.)

Observe carefully the value of the counterpoise K necessary to balance the roller.

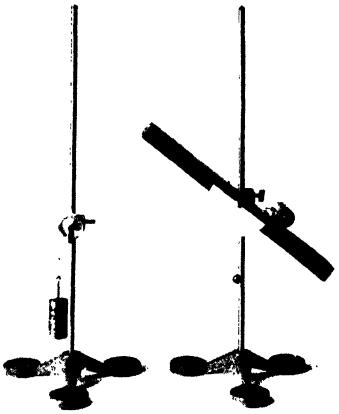

FIG. 10*b*.

To what error is your value of K liable, which was not present in EXPERIMENT X., and how may it be eliminated?

From theoretical considerations, what fraction of W do you expect K to be?

EXPERIMENT XI.

Object of Experiment:

To show that, in order to produce equilibrium, two couples* acting upon a body must be of equal moment, the one clockwise and the other anticlockwise in direction.

Apparatus:

Capstan stand, 4 Capstan blocks, 2 pulleys, pivot, drilled metre rule, 2 hangers, 3 metal rods, 2 lb. spring-balance, small retort-stand, ½-lb. ring-weight, slotted weights, set-square, short ruler, small piece of copper wire.

Method:

(1) The Capstan stand is to hold four Capstan blocks, as in fig. 11. The bottom block holds the pulley M, the next the pivot for the rule, the next the pulley L, and the top one the metal rod R from which the spring-balance hangs.

Adjust the heights of the two pulleys so that their highest points are each 30 cm. from O.

(2) Hang the rule on the pivot by the centre hole, and, if necessary, adjust it to balance horizontally with the aid of the small piece of copper wire K (fig. 11).

Place the small retort-stand with the two bars projecting from it, one just below, and the other just above one end of the rule, to prevent it swinging right up or down.

(3) Tie a small loop of thread about 3 in. long through the ring in the ½-lb. weight, and slip it over the end A of the rule to the 10 cm. scale reading—the point X. Then $F'_1 = \frac{1}{2}$ lb. wt.

* In the same plane.

Make a similar loop and slip it over the end B to the 90 cm. scale reading—the point Y. Attach to this the spring-balance hanging from the rod R, taking care that the connecting thread is accurately vertical. Let F_2 lbs. wt. = reading of spring-balance.

Then $x = y = 40$ cm.

(4) Attach a piece of thread by a loose loop to the hole at the 5 cm. scale reading (the point C), pass it over the pulley L, and attach a hanger to it. Let F_3 lbs. wt. = final weight on this hanger.

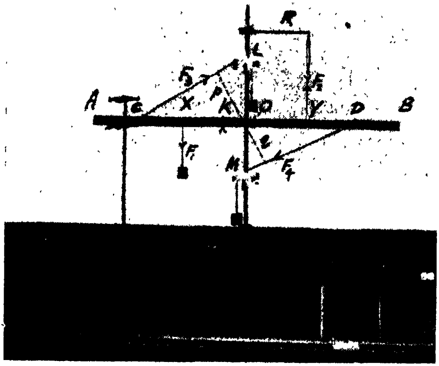

Fig. 11.

Similarly attach a thread to the hole at the 95 cm. scale reading (the point D), pass it over the pulley M, and attach a hanger to it. Let F_4 lbs. wt. = final weight on this hanger.

(5) Increase the weights on the two hangers by steps of ·01 lb. until the rule balances horizontally.

(6) Clamp the rule in this position of equilibrium by means of the two rods projecting from the small retort-stand, and measure, by the method of Experiment II. (4), the lengths of p and q, the perpendiculars from O upon LC and MD.

Record your observations in a table similar to Table XI. below, *working out your calculations at the time.*

(7) Repeat the above operations with $x = y$ successively 35, 30, 25 and 20 cm.; in each case varying the equal distances OL, OM.

TABLE XI.

Experimental Results.

Experiment No.	1	2, etc.
F'_1		
x		
F'_2		
y		
F'_3		
p		
F'_4		
q		
$F'_1x + F'_2y$		
Percentage diff. from mean		
$F'_3p + F'_4q$		
Percentage diff. from mean		
Mean		

EXPERIMENT XII.

Object of Experiment:

To determine the coefficient of friction between a given pair of materials; and to examine how this coefficient varies with the magnitude of the frictional forces called into play, and with the extent of the surfaces in contact.

Apparatus:

Oak or mahogany plane fitted with levelling screws and a pulley; oak and mahogany blocks, plain on one side, grooved on the other; slotted weights and hanger; thread; other weights $\frac{1}{4}$ to 2 lbs. by steps of $\frac{1}{4}$ lb.; balance; spirit-level; large stand fitted with platform.

Method:

(1) Place the plane (fig. 12*a*) at an end of the bench with the pulley projecting over the edge. Adjust the plane so that it is horizontal, both longitudinally and transversely, testing it with a spirit-level.*

(2) Weigh the block on a balance correct to ·01 lb.

(3) Place the block on the plane with the grooved † side upwards. Attach a thread to the block, pass it over the pulley, and attach a hanger to its other end. The platform should be adjusted at such a height that it receives the hanger when the block has moved a few inches along the plane.

(4) By piling weights on it, make up the effective weight of the block to $\frac{1}{2}$ lb.

Now add weights to the hanger by steps of ·01 lb. until the block will just move slowly when the plane is tapped gently.

---

* An alternative method of performing the experiment is shown in fig. 12*b*.

† The groove is shown in the annexe to fig. 12*a*.

You should not try to get the block to slide along the whole length of the plane: it is better to confine attention to a short length of it—say 2 in.—starting from a fixed point, and adjust for sliding over that portion only.

A really exhaustive determination demands that a separate set of readings should be taken for every few inches of the plane, and the mean value of the coefficient of friction for each set recorded. This, however, need not be attempted unless time allows.

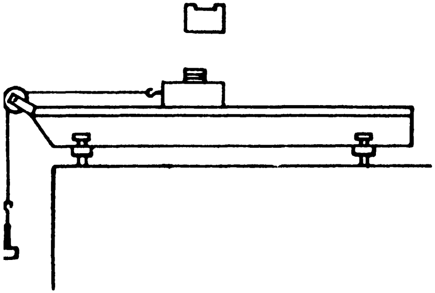

Fig. 12*a*.

(5) Place another $\frac{1}{4}$ lb. on the block, and add further weights to the hanger until the block moves slowly over the same part of the plane.

(6) Repeat the experiment, in each case increasing the weight on the block by $\frac{1}{4}$ lb., up to 2 lbs.

(7) Tabulate your results as in Table XII., working out the calculations *at the time*.

(8) Plot a graph of F against W (see Table XII.). What do you notice about the nature of this graph; and what do you infer from it?

(9) Now reverse the block, so that the grooved side is downwards, and repeat all the above operations. Do you get the same value of μ as for the smooth side? What do you infer from your results?

TABLE XII.

Experimental Results.

Number of Experiment.	1	2, etc.
Materials of which plane and block are composed		
Effective weight of block (= W lbs. wt.)		
Force which produces slow motion of block (= F' lbs. wt.)		
Coefficient of friction μ (= F'/W)		
Mean value of μ =		

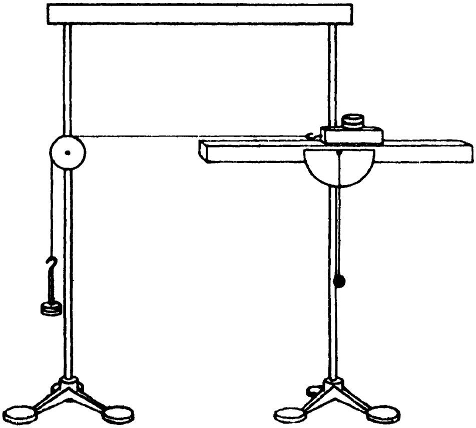

FIG. 12b.—Alternative form of apparatus.

Experimental Exercise.

Wrap a piece of paper round the plane, and another round the wood block, securing them with drawing-pins. Hence determine μ for one paper surface sliding over another.

The experiment may be varied by using different kinds of paper.

EXPERIMENT XIII.

CENTROID OF A LAMINA.

Object of Experiment:

To show that the centroid of a triangular lamina ABC is at the point of intersection of the medians of the triangle.

Apparatus:

Cardboard triangular lamina, Capstan stand and block, pivot, thread, plumb-bob, cork-borer, gum, geometrical instruments, pin.

Method:

(1) Cut a piece of paper exactly the shape of the triangular lamina. With geometrical instruments draw carefully on the paper the medians of the triangle, which, of course, should be concurrent.

(2) Bore holes in the paper opposite those in the cardboard, and fix it lightly to the cardboard with a few spots of gum, the blank side of the paper being uppermost.

(3) Hang the cardboard up by the vertex A from the pivot (fig. 13*a*). Knot a small loop on the thread of the plumb-line and hang it from the pivot in front of the cardboard.

With a set-square and fine pencil mark two points perpendicularly behind the thread of the plumb-line.

When the cardboard is subsequently removed these two points are to be joined by a straight line.

Since the lamina is in equilibrium in this position, its centroid G must be somewhere on this line.

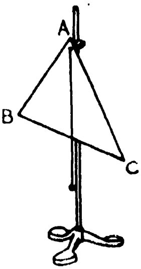

FIG. 13*a*.

(4) Hang the lamina up by the vertex B and proceed exactly as in (3).

The centroid G must evidently be at the point of intersection of the two lines obtained.

(5) As a check on the accuracy of the above operations, hang the lamina up by the vertex C and again proceed as in (3).

The third line thus obtained should also pass through G.

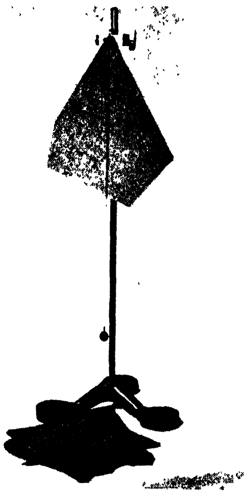

FIG. 13*b*.

(6) With the pin prick a hole in the paper at the point G. Now remove the paper, and on the reverse side measure the distance of the pin hole from the point of concurrence of the medians.

Experimental Exercise.

Find by a similar method the centroid of a polygonal lamina. As in the last Experiment, check by a geometrical method on the reverse side of the paper (fig. 13*b*).

* EXPERIMENT XIV.

CENTROID OF A SOLID BODY.

Outline of Method :

This Experiment forms a convenient method of finding the position of the centroid of an irregularly-shaped body of large size—e.g. an aeroplane—for which the method of the last Experiment (suspension of the body from two different points) is inconvenient.

The method consists in mounting the body on a horizontal platform suspended by strings or chains from a single point overhead (fig. 14*a*). From the platform projects a horizontal rod of suitable length (AB in fig. 14*b*). A weight of known magnitude, small compared with the weight of the body, is hung on this rod at a measured distance from the vertical through the point of suspension.

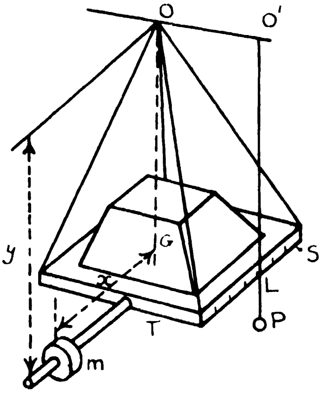

FIG. 14*a*.

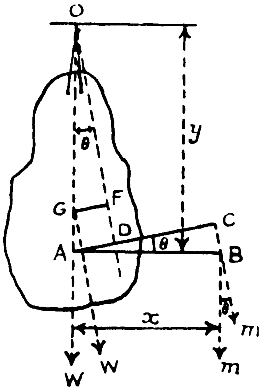

FIG. 14*b*.

The consequent displacement of the platform from its previous position—measured by a plumb-line and a scale attached to the platform—provides data from which the position of the centroid of the body may be calculated.

Theoretical.

Let G be the centroid, and W lbs. wt. the weight of the body.

Attach a weight m lbs. wt. to the body at a point B, whose co-ordinates measured from O (the point of suspension) are (x, y), in the vertical plane through O.

This causes the system to be displaced, in this plane, through an angle θ (which will be small, provided that the ratio $m : W$ is small).

Let OF (fig. 14*b*.) be the line which now becomes vertical, making an angle θ with OG.

Draw GF and ADC perpendicular to OF, BA and BC perpendicular to OG and AC, respectively.

Now from the figure we evidently have

$$\begin{aligned} \mathrm{GF} &= \mathrm{OG}\,.\sin\theta \\ \mathrm{AC} &= x\,.\cos\theta \\ \mathrm{AD} &= y\,.\sin\theta. \end{aligned}$$

Then, taking moments about O,

$$W.\mathrm{GF} = m\,.\mathrm{DC},$$

or
$$W.\mathrm{OG}\sin\theta = m\,(\mathrm{AC} - \mathrm{AD}),$$

i.e.
$$W.\mathrm{OG}\sin\theta = m\,(x\,.\cos\theta - y\,.\sin\theta).$$

Thus
$$\mathrm{OG} = m\,(x\,.\cot\theta - y)/W,$$

an equation which gives the vertical distance of G from O in terms of experimentally determinate quantities.

Note. —If, as is usually the case, the deflection be small

$$x\,.\cot\theta = x/\theta \text{ (if } \theta \text{ be measured in radians)},$$

and the equation for OG becomes

$$\mathrm{OG} = m\,(x/\theta\,. - y)/W.$$

Experimental.

Place the body on the platform T (fig. 14*a*) in such a way that the platform is horizontal.

Attach a scale S at the side of the platform, as in the figure;

and suspend the plumb-line O′LP so that it just touches the scale, O′ being at the same vertical height as O, and OO′ being perpendicular to the vertical plane in which x and y are measured.

Let L be the point at which the plumb-line touches the scale,
d = deflection as read on the scale against the plumb-line when the weight m is added.

Then, provided $d/\text{O}'\text{L}$ is small,

$$\theta = d/\text{O}'\text{L} = d/y.$$

The following are the results of an experiment:—

W = 115 lbs. wt., d = 1·25 in.,
m = 5 lbs. wt., O′L = y = 42 in.,
x = 20 in.,
y = 42 in., so θ = 1·25/42 radian.

Hence OG = 5(840/1·25 − 42)/115 = 27·4 inches.

Thus the height of the centroid from the base is

$$42 - 27·4 = 14·6 \text{ inches}.$$

The arrangement of apparatus for this experiment has been left as an exercise for the student. If, as is suggested, the platform be suspended from two Capstan stands, smaller values of m and W should be chosen.

Problem on Experiment XIV.

In the Theoretical treatment of the above experiment, the weight of the platform has been neglected. Explain how you would determine the percentage error in the value of OG caused by neglecting the weight of the platform.

EXPERIMENT XV.

THE WHEEL AND AXLE.

Object of Experiment:

(*a*) **To find the *velocity-ratio* (*V*)[*] of the machine.**

(*b*) **To find the *efficiency* (*e*)[†] of the machine for various loads.**

(*c*) **To find the *law of the Machine,* i.e. to test whether the "effort"[‡] *P* and the load *L* satisfy a relation of the form**

$$P = a + bL,$$

where a and b are constants whose values are to be determined.

Apparatus:

Wheel-and-axle, 2 Capstan stands, 2 Capstan blocks, ¼, ½, 1, 2, 4-*lb. ring weights, small brass hook,* 1/10-*lb. hanger and slotted weights, small piece of copper wire (or lead foil) for counterpoise,* 2 ½-*metre rules graduated in millimetres,* 2 *small retort-stands, two* 2-*lb. weights to place on these.*

Procedure:

(*a*) ***Velocity-ratio* (*V*).**

(1) Set up the apparatus as shown in Figs. 15*a* and 15*b*.

In Fig. 15*a* it is shown that two ½-metre rules are to be set up by the side of the effort and the load respectively.

(2) Pull down the effort string so that the load rises 2 cm. exactly.

Observe carefully the distance which the effort string has descended in this operation. Half of this will be, numerically, the velocity-ratio.

---

[*] The *velocity-ratio* of the machine is the distance the effort moves while the load rises unit distance.

More generally, it is the ratio of the distance the effort moves to that which the load ascends.

[†] The *efficiency* of the machine is the ratio of the work done on the load to that done by the effort.

[‡] That is, the *applied force*.

(3) Let the load down gently on to the table, and repeat operation (2), but raising the load 4 cm. In this case the distance which the effort string descends must be divided by 4 to give the velocity-ratio.

(4) Repeat operation (3) as many times as possible, each time increasing the distance the load ascends by 2 cm.

(5) Take the mean of all the above values of V as the value to be used in part (*b*) of the experiment.

All readings should be recorded in tabular form, and the calculations worked out *at the time*.

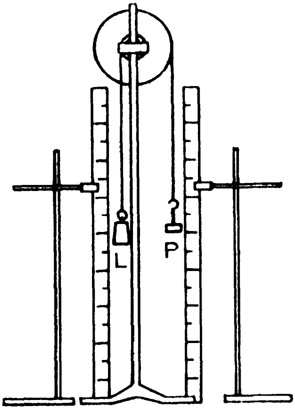

Fig. 15*a*.

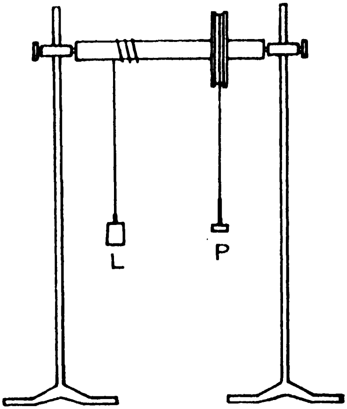

Fig. 15*b*.

Table XVa, which gives the result of an actual experiment, shows a convenient way of tabulating the results:—

TABLE XVa.

Observations of Velocity-ratio (V).

Observation No....	1	2	3	4, etc.
Load ascends	2 cm.	4	6	8
Effort descends ...	7·25 cm.	14	22·2	
V	2·63	3·5	3·7	

Mean value of $V = 3·61$.

(*b*) ***Efficiency*** (*e*).

Theoretical.

Let L (lbs. wt.) = Load,
P ,, = effort necessary just to raise the load.

In ascending x feet, the work done on the load is

$$Lx \text{ ft.-lbs.} = W_1 \text{ (say)}.$$

Meanwhile the effort descends a distance

$$Vx \text{ feet},$$

and the work done by the effort is

$$PVx \text{ ft.-lbs.} = W_2 \text{ (say)}.$$

Since some of this work is wasted in overcoming the friction of the machine, W_1 is less than W_2,

put $W_1 = eW_2$ (e = efficiency of machine),

then $e = W_1/W_2 = Lx/PVx$,

i.e. $e = L/PV$.

Experimental.

(1) Attach a small hook to the string wound round the axle (figs. 15*a*, 15*b*). In order that the weight of the hook may not enter into the calculations, attach to the effort string a small piece of copper wire, or lead foil, whose size should be adjusted so that it just balances the hook when neither strings are loaded.

(2) Attach a ½-lb. ring weight to the hook.

(3) Make a loop on the effort string, and attach to it a 0·1 lb. hanger.

Add weights in steps of 0·01 lb. until the effort string just begins to descend.

(4) Record the values of the load L and the effort P in a table similar to Table XV*b*. below.

Hence calculate the value of e.

(The calculation must be made *at the time*. This is a check on the accuracy of the observations, for if they yield an impossible value of e, e.g. greater than unity, a mistake in reading must have been made. The observations should then be rejected and a fresh set taken.)

(5) Perform successive experiments similar to the above, in each case increasing the load by ½ lb.

Table XV*b*., which gives the results of some actual experiments, shows a convenient method of recording results.

TABLE XVb.

Observations of the Efficiency.

Observation No....	1	2	3	4	5	6	7
L	0·5	1·0	1·5	2·0	2·5	3·0	4·0
P	0·2	0·33	0·42	0·6	0·73	0·89	1·13
$e = L/PV$ (%) ...	69·3	84·0	99·0	92·3	94·8	93·4	98·0

(*c*) ***Law of the Machine.***

(1) It is surmised that P and L are connected by a relation of the form

$$P = a + bL,$$

where a and b are constants.

If this surmise is correct, the graph of P plotted against L will be a straight line.

(2) From your readings, as above, plot a graph of P against L. Actually it will be found that the points so obtained do not lie exactly on a straight line. In this case, a straight line should be drawn which lies evenly between the points, as in fig. 15*c*.

Determination of the values of the constants a and b.*

Produce this straight line downwards to meet the P-axis in X, and the L-axis in Y.

Let O be the point of intersection of the P- and L-axes (fig. 15*c*.)

Then a = OX = **0·065** (from the figure); OY = 0·24

and b = OX/OY (numerically) = 0·065/0·24 = **0·27**.

---

* This method is given as being the simplest. A more accurate method, when the line passes close to the origin, is given in Experiment XVI., Section II.

Of the seven points obtained, it will be seen that five lie closely upon the straight line XY. The deviations of the other two must be set down to observational errors.

We are thus entitled to infer that our surmise has been correct; that there is a linear relation between P and L, and that the Law of this Machine is

$$P = 0{\cdot}065 + 0{\cdot}27\,L.$$

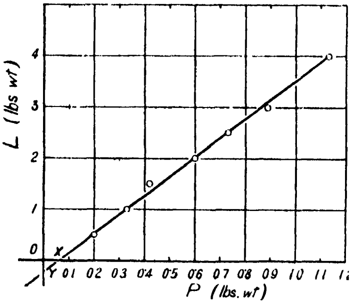

FIG. 15c.

Exercises.

1. What physical significance has the term a in the equation for the Law of the Machine?
2. Calculate the values of a and b for the observations in Table XVb., by the method referred to in the footnote.
3. Plot a graph showing how the efficiency varies with the load.
4. If the effort corresponding to a given load be underestimated, what effect has this on the calculated value of the efficiency?
5. If D be the diameter of the wheel, and d that of the axle, calculate the velocity-ratio. Use this to check the experimental value you obtain.

EXPERIMENT XVI.

THE COMPOUND PULLEY.

Object of Experiment:

(*a*) **To find the** *velocity-ratio* (*V*) * **of the machine.**

(*b*) **To find the** *efficiency* (*e*) * **of the machine for various loads.**

(*c*) **To find the** *law of the Machine,* **i.e., to test whether the effort** P **and the load** L **satisfy a relation of the form**

$$P = a + bL,$$

where a **and** b **are constants whose values are to be determined.**

Two forms of apparatus have been found to be convenient:

Set No. 1.

A small-size apparatus in which the pulley wheels are the size of that used in EXPERIMENT II., spring-balances being used for the effort and the load. A particularly convenient form of pulley-block for this purpose has been made by Messrs. Cussons. In each block the wheels are of equal size and are arranged in echelon, with the object of showing all the strings easily (fig. 16*a*).

An apparatus of this kind has the advantage that it is easily set up: also no great experimental skill is required in order to obtain good readings.

Set No. 2.

A large compound pulley of commercial size (fig. 16*c*). While more cumbersome, and less capable of yielding accurate readings than the former type, this has the advantage of introducing the student to a practical problem.

---

* For definitions of velocity-ratio and efficiency, see EXPERIMENT XV., p. 46, footnote.

SECTION I.

Theoretical Treatment of Velocity-Ratio and Efficiency.

Velocity-ratio.—Let n be the total number of pulleys on the two blocks. If n be even, there will be the same number on each block; if n be odd, there will be one more on the upper block.

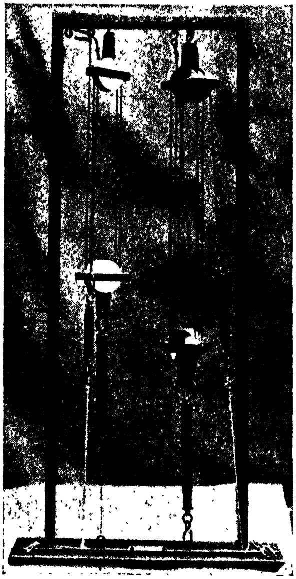

FIG. 16*a*.—Echelon pulleys.

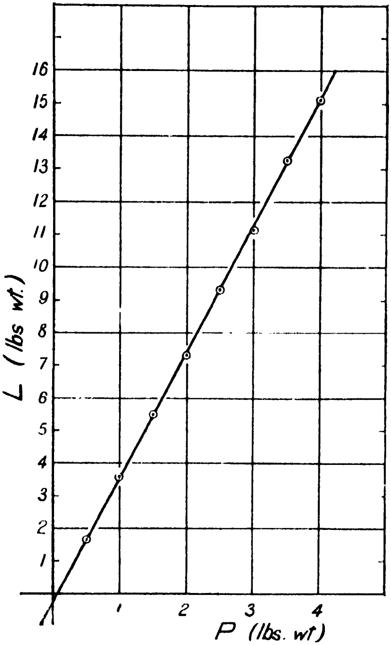

FIG. 16*b*.

In every case, the number of strings—all portions of the same string—which support the load is n.

On pulling down the effort string so that the load rises 1 foot, each of these strings is shortened by 1 foot. The total shortening is thus n feet, and so the effort string descends n feet.

Therefore velocity-ratio $V = n$.

Efficiency.—Let L lbs. wt = load,

P „ „ = effort necessary just to raise the load.

Then, by an argument* similar to that employed in EXPERIMENT XV. (*b*) p. 48, we see that

$$e = L/nP.$$

SECTION II.

Experiment with Echelon Pulley.

The experiment is a simple one to perform, and the procedure is left to the initiative of the student; the following points, however, should not be neglected.

(1) Before using the spring-balances, test them by hanging known weights on them, to see if the graduations are correct. Notice also it the pointer is accurately at the zero graduation when there is no load on the balance. (These remarks apply equally to all experiments with spring-balances.)

(2) The theoretical value of the velocity-ratio should be tested experimentally by the method used in EXPERIMENT XV.

Details of an experiment:—

The following readings were obtained with a 4-sheaf echelon pulley ($n = 4$).

P	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
L	1·65	3·6	5·5	7·3	9·33	11·14	13·25	15·1

The points corresponding to these readings are plotted in fig. 16*b*. It will be seen that the points lie sufficiently nearly upon a straight line for us to assume a linear relation between L and P; that is, we are justified in assuming an equation of the type

$$a + bL = P$$

as the "Law of the Machine."

We now draw the straight line to which these points seem most closely to approximate; and proceed to determine its equation, that is, to find the values of a and b in the equation above.

It will be noticed that the straight line in fig. 16*b* passes very close to the origin. As the intercepts are thus small, the method used in EXPERIMENT XV. will not give reliable results, and we proceed as follows:—

---

* The student should write out for himself the proof as it applies to this particular experiment.

Taking the second and last readings, we have the two pairs of values (which lie on the graph),

$$\begin{aligned} P &= 1{\cdot}0 \\ L &= 3{\cdot}6 \end{aligned} \quad \text{and} \quad \begin{aligned} P &= 4{\cdot}0 \\ L &= 15{\cdot}1. \end{aligned}$$

Substituting in the above equation, we have :—

$$\begin{array}{ll} & a + 3{\cdot}6\,b = 1, \\ \text{and} & a + 15{\cdot}1\,b = 4, \\ \text{whence} & 11{\cdot}5\,b = 3, \\ \text{and} & \mathbf{b = 0{\cdot}26.} \\ \text{So} & a = 1 - 3{\cdot}6 \times 0{\cdot}26, \\ \text{that is} & \mathbf{a = 0{\cdot}06.} \end{array}$$

Thus the Law of the Machine is

$$\mathbf{P = 0{\cdot}06 + 0{\cdot}26\,L.}$$

SECTION III.

Experimental Details for Set II. Apparatus.

Apparatus :

Commercial size compound pulley (such as that shown in fig. 16c which has 3 pulleys on each block) ; large kilogramme hanger and slotted weights up to 20 kgs. ; kilogramme spring-balance reading to 10 kgs. in tenths ; small coconut mat to place under weights.*

It is convenient to hang the pulley from a hook in the ceiling by a length of ¼-in. iron rod bent into a hook at each end, the rod to terminate about 6 ft. 6 ins. from the floor. Where this is not practicable a stout wooden beam of length about 7 ft., fitted at its ends with wooden feet, and at the centre with a strong hook, and mounted on stools, or across the tops of two cupboards at the angle of the room, may be used instead.

Observations for the Efficiency at Various Loads.

(1) Hang the pulley from the hook, and attach to its lower end a kilogramme hanger.

(2) Place a stool with the coconut mat upon it vertically below

---

* No apology is made for employing metric units here, while British units were used in the theoretical treatment. It may often happen that a Laboratory is not furnished with large weights of this type in both lbs. and kgs.

the hanger, and adjust the effort rope until the hanger is just let down on to the mat.

Make up the total load to 10 kgs.

(3) In order to keep the system in this position while making preparations, knot a loop on the effort rope on a level with the hook of the hanger.

Slip this loop temporarily over the hook of the hanger. This will prevent the lower block from running down, if the apparatus has to be left for a moment.

(4) With the lower block in this position, knot another loop on the effort rope about the level of the eye.

Attach the spring-balance to this loop, and disconnect the lower loop from the hanger.

(5) Attempt to raise the load by pulling down the effort rope with the spring-balance.

Observe the reading of the spring-balance at the instant when the load begins to *ascend*. Commercial pulleys being roughly constructed, it will probably be found that this spring-balance reading is not sharply defined.

After obtaining this reading, gradually relax the pull on the spring-balance until the load just begins to *descend*, and again note the reading.

This pair of observations should be repeated 4 times more, and the mean of the 5 pairs taken as the value of P, the effort required just to raise a load of 10 kgs.

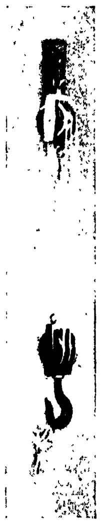

FIG. 16c.

(6) Repeat operation (5) with a load of 11 kgs.; and similarly, in steps of 1 kg., up to a final load of 20 kgs.

Make a table of your readings with columns for L, P and e. The observations should be entered in this table *at the time.*

(7) **Law of the Machine.**—Test, as in Section II., whether a relation between P and L of the form

$$P = a + bL$$

is applicable to this machine.

EXERCISES ON EXPERIMENT XVI.

1. Check experimentally the value of the velocity-ratio for either set of apparatus, by the method used in EXPERIMENT XV.
2. Using Set II. apparatus, weigh the empty lower block with a spring-balance. Hence calculate what fraction of the wasted work is employed in

   (i) raising the weight of the lower block;
   (ii) overcoming friction.

3. Calculate the values of the efficiency *e* for the table of readings given in Section II.
4. Check the values of *a* and *b* obtained in Section II. by substituting other values of *L* and *P* in the equation for the law of the Machine.
5. What inferences as to the mode of variation of the efficiency with the load may be drawn from the *L* and *P* graph?

EXPERIMENT XVII.

THE DIFFERENTIAL PULLEY.

Object of Experiment:

(*a*) **To find the** *velocity-ratio* * **(*V*).**

(*b*) **To find the** *efficiency* * **(*e*) of the machine for various loads.**

(*c*) **To find the** *law of the Machine*, **i.e. to test whether the effort P and the load L satisfy a relation of the form**

$$P = a + bL,$$

where a and b are constants whose values are to be determined.

Apparatus:

Differential pulley, long iron rod hung from a hook in the ceiling (or alternatively, as in EXPERIMENT XVI.), large kilogramme hanger with slotted kg. weights up to 20 kgs., kg. spring-balance reading up to 10 kgs. in tenths, small coconut mat, duster, small piece of strong string.

Method:

(*a*) *Velocity-ratio.*

Theoretical.

Let L kgs. wt. = load,

P ,, = effort necessary just to raise the load,

t_1, t_2 = no. of teeth on large and small wheel of upper block, respectively,

x cm. = distance between two consecutive teeth on either wheel of the upper block (the actual distance need not be measured).

In one revolution of the upper wheel-pair, the effort descends a distance

$$t_1 x \text{ cm.}$$

---

* For definitions of velocity ratio and efficiency, see EXPERIMENT XV., p. 46, footnote.

But the chain is also wound up on the smaller wheel an amount

$$t_2 x \text{ cm.}$$

The chain supporting the load is thus shortened an amount

$$(t_1 - t_2) . x \text{ cm.}$$

Now this shortening must distribute itself equally on either side of the lower pulley.

Hence the load rises a distance

$$\tfrac{1}{2} (t_1 - t_2) . x \text{ cm.}$$

Thus the velocity-ratio

$$V = t_1 x / \tfrac{1}{2}(t_1 - t_2) . x$$

$$\text{or } V = 2t_1/(t_1 - t_2).$$

Experimental.

Count the number of teeth on each wheel of the upper block.

By substituting these values in the formula just obtained, calculate the numerical value of V.

Use this value in Section (*b*) below.

(*b*) *Efficiency.*

Theoretical.

By an argument exactly similar to that in EXPERIMENT XV., p. 48, it may be proved that the efficiency

$$e = L/VP.$$

Experimental.

(1) Set up the apparatus as in fig. 17. The chain and wheel axles should be oiled before use, in order to diminish friction as much as possible.

To avoid soiling the hands, the chain should be gripped with a duster when raising or lowering the load.

(2) Attach a hanger with a total load of 10 kgs. on to the hook under the lower pulley.

Since the machine does not "overhaul" (see note on p. 60 below), it is unnecessary to secure the effort chain when leaving the instrument.

(3) Raise the load, and put underneath it a stool with the coconut mat on it.

Lower the load so that it just rests on the mat.

(4) Tie a loop of strong string into one of the links of the power chain, about the level of the eye. Hook the spring-balance on to this loop.

By pulling vertically downwards on the spring-balance, attempt to raise the load. Note the reading of the spring-balance when the pull is just sufficient to cause the effort chain to move down slowly.

As a commercial pulley of this kind is roughly constructed, and so inclined to move in jerks, the above observation should be repeated ten times. If the mean of the first five of these differs sensibly from that of the whole ten, further observations should be made until a consistent mean is obtained.

(5) Repeat operation (4) with loads of 11, 12, etc., up to 20 kg.

(6) Calculate the efficiency e from the formula given above, and enter the values of L, P and e in a table similar to Table XVII. below.

It is important that the calculations should be made *at the time*, as this affords a valuable check on observational inaccuracies.

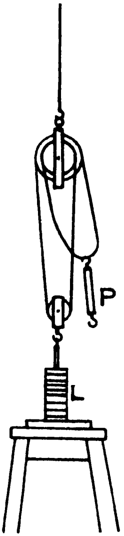

FIG. 17.

TABLE XVII.

Observations of L, P, and e.*

Observation No.	1	2	3, etc., to	11
Load L (kgs. wt.)	10	11	12	20
Effort P (kgs. wt.) (Mean of 10 readgs.)† ...	4·6	5·0	5·4	8·18
Efficiency e (%)	12·1	12·2	12·35	13·6

No. of teeth on larger wheel $t_1 = 9$,
" smaller " $t_2 = 8$,
Velocity-ratio $V = 18$.

---

* A few readings only are given here. The student should, of course, make a complete table of his own readings.

† If a different number of readings was taken to obtain the mean, state this number in the table.

(c) *Law of the Machine.*

The relation $P = a + bL$

may be tested as in EXPERIMENT XV. (c), p. 49.

Note on "Overhauling."

A machine is said to "overhaul" or "take charge" if it permits the load to run down when the effort is removed.

One of the advantages of the differential pulley over the compound pulley is that it does not overhaul.

It may be shown that the efficiency of a machine which will not overhaul cannot exceed 50 per cent.

Suppose, for example, that the efficiency were 40 per cent.: then for every 100 cm.-kg. of work done by the effort, 40 cm.-kg. are employed in raising the load, and 60 cm.-kg. in overcoming friction, etc.

If, now, the load were to run down under its own weight the same distance as it has been raised, it would do work to the extent of 40 cm.-kg.

But, in order to move, it must do nearly 60 cm.-kg. of work in overcoming the resistance of the machine.

It is thus unable to move of its own accord.

* EXPERIMENT XVIII.

THE SCREW.

Object of Experiment:

(*a*) **To find the *velocity-ratio* (V) * of the machine.**

(*b*) **To find the *efficiency* (e) * of the machine for various loads.**

(*c*) **To find the *law of the Machine*, i.e. to test whether the effort P and the load L satisfy a relation of the form**

$$P = a + bL,$$

where a and b are constants whose values are to be determined.

Apparatus:

Two Capstan stands, screw in holder, Capstan block and pulley, wooden tie beam, $\frac{1}{2}$, 1, 2, 4-*lb. ring-weights,* 1/20 *lb. hanger and slotted weights in steps of* 1/100 *lb., small brass hook, stout thread (or thin string).*

Description of Apparatus:

The apparatus (fig. 18) consists of a wooden wheel H with a brass axle which is tapped with a screw thread, and runs in a brass block attachable to a Capstan stand.

The effort acts at the circumference of the wheel, while the effective load is moved in the direction of advance of the screw.

Setting up of Apparatus.

(1) The screw-holder should be clamped to the Capstan stand as near the top as possible, to allow a long run of the effort string when measuring the velocity-ratio.

---

* For definition of velocity-ratio and efficiency, see EXPERIMENT XV., p. 46, footnote.

(2) Attach a piece of thin string to the ring at the end of the screw, and pass it over a pulley which is to be fixed by a Capstan block to the second stand.

(3) Fit the wooden tie-beam over the tops of the two stands to prevent them approaching each other.

(4) Carefully adjust the height of the pulley so that the string is accurately along the axis of the screw.

(5) Make a loop on the end of the string about 8 inches below the pulley, and attach the hook to it.

Attach a small ring-weight to the hook to keep the string taut while making final adjustments.

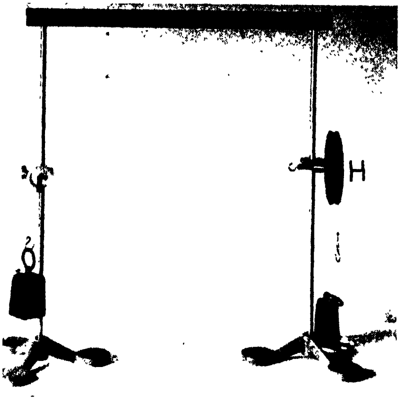

FIG. 15.

(6) Adjust the "azimuth" of the Capstan block holding the pulley so that the string does not rub against the groove on either side of the pulley.

(a) *Velocity-ratio.*

Theoretical.

Let p inches = pitch of screw (the distance—measured axially—between consecutive threads).

d „ = diameter of wheel on circumference of which effort acts.

In one revolution the effort descends a distance

$$\pi d \text{ inches}$$

while the load ascends a distance

$$p \text{ inches.}$$

Hence $$V = \pi d/p.$$

Experimental.

The velocity-ratio may be determined experimentally exactly as in EXPERIMENT XV. (*a*), p. 46.

The details are left as an exercise for the student.

(*b*) *Efficiency.*

Theoretical.

By an argument similar to that in EXPERIMENT XV. (*b*), p. 48, it may be shown that the efficiency e of the machine is given by

$$e = L/VP,$$

where V is the value of the velocity-ratio as already determined.

Experimental.

(1) Hang a 2-lb. weight on the hook at the end of the string.

(2) Make a loop on the end of the string passing round the wheel, and attach the 1/20 lb. hanger to it.

(3) Add weights in steps of 1/100 lb. until the effort string begins to move down *slowly*.

The observation should be repeated several times for the same load in order to secure as accurately as possible that the effort is the smallest which will just give motion.

(4) Increase the load to 2·5 lbs., and repeat operations (1) to (3).

(5) Perform similar operations with loads of 3, 3·5, etc., in steps of ½ lb. up to 7 lbs.

Record your results in a table similar to Table XVIII. below.

TABLE XVIII.

Observations of L, P, and e.

Observation No.	1	2	3, etc., to	11
L (lbs. wt.)	2	2·5	3	7
P (lbs. wt.)				
$e = L/PV$ (°/$_o$)				

(c) *Law of the Machine.*

The relation

$$P = a + bL$$

may be tested exactly as in EXPERIMENT XV. (*c*), p. 49.

* EXPERIMENT XIX.

HOOKE'S LAW AND YOUNG'S MODULUS.

Object of Experiment: †

(*a*) **To show that if a wire is extended by a gradually increasing load, then, provided the elastic limit is not reached, the extension produced is proportional to the load (Hooke's Law).**

(*b*) **To show that if the load be reduced by the same stages as those by which it was increased, the wire regains the length it previously had; and that on the removal of all the load, the wire regains its original unstretched length.**

(*c*) **To show that if two wires of the same material and thickness, but of different lengths, be employed, the longer one is extended more by a given load than the shorter one; and to find the relation between the relative lengths and the relative extensions.**

(*d*) **To show that if two wires of the same material and length, but of different thicknesses, be employed, the thicker one is extended less by a given load than the thinner; and to find the relation between the relative thicknesses (areas of cross-section) and the relative extensions.**

(*e*) **To find the value of Young's Modulus for the material of the wire.**

Apparatus:

Two pieces of steel wire 2 metres long; another pair of the same length, but of different thickness; vernier and scale; stout wooden beam about 6 ft. long to which is attached a special metal clamp to secure the upper ends of the wires; Capstan stand and block; pivot for ditto; drilled metre rule; 1 kg. hanger and slotted weights; 3 weights to place on the feet of the Capstan stands; metal ring to keep the lower ends of the unloaded wires taut; micrometer gauge.

---

† Before doing this experiment the student should read Note 1 of Appendix II.

First Method:

Fit up the apparatus as shown in fig. 19*a*. By using a metre rule (drilled every 5 cm.) as a lever, and a kg. hanger, an effective load varying from 0 up to 18 kg. may be applied to the wire by merely sliding the hanger H along CB.

The hanger should be suspended from the rule by a loose loop of thin string.

The vernier V should be attached to the wire CD which is to be stretched, and the scale S to the other wires EF, E′F′.

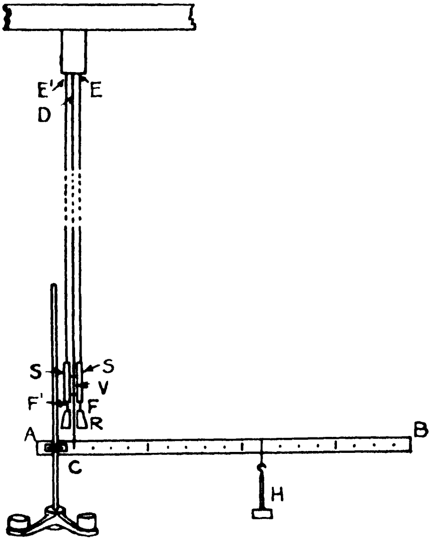

Fig. 19*a*.

The lower ends F, F′ of these wires should be attached to the metal ring R, to keep them taut. This is shown more clearly in fig. 19*b*, where the vernier arrangement is shown unconnected to the lever AB of fig. 19*a*.

The upper ends D, E, and E′ of the wires are firmly held in the special clamp (fig. 19*b*), which is screwed to the wooden beam.

To prevent the Capstan stand moving, a weight—say 5 kg.—should be placed on each foot.

Procedure:

To test (*a*) **and** (*b*) :—

(1) Take the thinner wire for CD, the wire to be experimented upon, and attach it as described above.

(2) Adjust the height of the Capstan block so that the end B of the rule is slightly higher than A.

(3) Measure carefully the length of the wire from the point D to the zero on the vernier; also measure its diameter *d* cm. with a micrometer gauge.

(4) By placing the hanger at the correct place on the rule, apply effective loads of 1, 2, 3, etc., up to 18 kg., in each case noting the extension produced.

(5) Now reduce the effective load in steps of 1 kg., by moving the hanger back along the rule; in each case noting the scale reading,—is it the same as for the corresponding load in (4)?

(6) Plot a graph of the extension against the load (*a*) for increasing loads; (*b*) for decreasing loads, on the same axes. Hence determine whether

(i) the extension is proportional to the load, and

(ii) whether the wire exhibits a "permanent set."

Record your readings in tabular form.

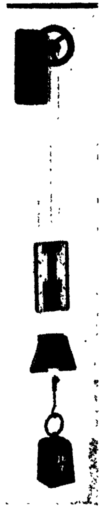

FIG. 19*b*.

To test (*c*) :—

(7) Reduce the effective length of the wire to exactly half that in (1); and repeat operations (2) to (6).

What conclusions do you draw from this second set of observations?

To test (*d*) :—

(8) Remove the wire CD, and replacing it by a thicker one of the same length as that used in (1), repeat operations (2) to (7).

To calculate (e) **Young's Modulus** :—

This may be calculated by selecting any single observation from the above, and computing from it the value of the strain s.

Now s = (total extension)/(unstretched length).

Then, if M kgs. wt. is the corresponding load, Young's Modulus Y is given by

$$Y = 4M/\pi d^2 s. \qquad \text{(kgs. wt. per sq. cm.)}$$

A more satisfactory way is to obtain from each of the four graphs above the average value of

load/extension;

and hence obtain a value of Y for each of the four sets of observations above.

Second Method :

A more sensitive, though more costly, instrument for performing these experiments is an extensometer designed by Searle* (figs. 19c, 19d). The figures are to a large extent self-explanatory.

The wire whose extension is to be observed is shown on the left. The other wire, which need not be of the same material, is kept taut by the metal block X.

Both wires, which should be of the same length, are gripped tightly at their ends by small 4-jaw chucks. The left-hand wire is extended by adding weights to the hanger H.

The method of measuring the extension requires some explanation.

The metal frames AB, CD, to which the lower ends of the wires are attached, are held apart by a rod (just above the spirit-level) which is freely pivoted to each of them.

A sensitive spirit-level, the tube of which is graduated, rests upon a cross-rod of the frame CD; its other support being the end of a micrometer-screw which works in the frame AB.

In order to measure the extension produced in the wire by a given load, the micrometer-screw is first adjusted so that the level is accurately horizontal, the scale-readings of the ends of the bubble, and of the micrometer being noted.

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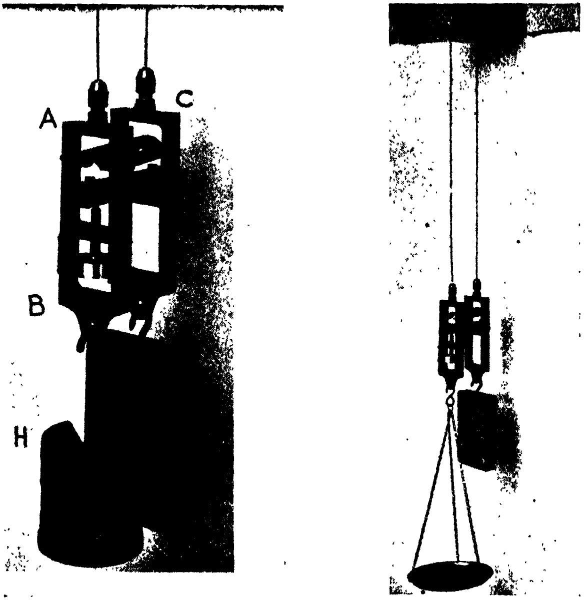
* G. F. C. Searle, *Proceedings of the Cambridge Philosophical Society* (1900), vol. x., p. 318.

The load is now placed on the hanger, and the consequent extension of the wire causes the level to tilt down on the right.

The spirit-level is now restored to its initial position by raising the micrometer-screw.

It is evident that the amount by which the wire has extended is equal to that by which the screw has ascended.



Figs. 19*c* and 19*d*.—Searle Extensometer.

It is to be noticed that the sensitiveness of the whole apparatus is limited by the sensitiveness of the spirit-level; this important part of the apparatus therefore calls for some consideration here.

A spirit-level consists, essentially, of a glass tube nearly filled with a spirit, such as alcohol, for which the surface-tension is low.

The glass tube containing the spirit is of uniform cross-section, and is bent, in the direction of its length, into the arc of a circle. On the degree of curvature of this arc depends the degree of sensitiveness of the level. In the ordinary carpenter's level the curvature is considerable; the bubble quickly takes up its steady position, but would show no deviation from this position for the minute angular displacements which our experiment involves.

It is important, then, that the tube of the level should be of small curvature, i.e., bent into a circle of large radius.

In the instrument used for the experiments detailed below, the tube of the level was of length 2 in., and contained ten graduations one tenth of an inch apart. The angular deflection per scale division was 1′ of arc.

The pitch of the micrometer screw was 1 millimetre, and the head was graduated in hundredths.

The student who uses this form of instrument should perform with it all the operations (1) to (8) indicated on p. 67 above.

Details are now given of an experiment performed with this instrument on a specimen of steel piano-wire.

In column 3 of Table XIX., the third decimal place is an estimated figure; columns 4 and 6 thus show that Hooke's Law is established for this wire with a variation of only seven-tenths of a micrometer division; that is, within the limit of accuracy imposed by the instrument.

The fact that the micrometer readings for decreasing load in this table are, at the end, slightly larger than at the outset may perhaps be due to "backlash" of the screw, which was not taken into account. In order to eliminate this error, all final adjustments of the micrometer should be made by approaching the desired position in one direction only; for example, by rotating the graduated head of the screw in the direction of increasing numbers. Thus for decreasing loads it is necessary to overshoot the mark and come back again in order that the reading may be approached in the required direction.

TABLE XIX.

Observations made on a Specimen of Steel Piano-wire with a Searle Pattern Extensometer, for the Testing of Hooke's Law and the Determination of Young's Modulus.

Load Increasing.				Load Decreasing.		
Obsn. No.	Load M (kgs. wt.)	Reading of Micrometer in mm.	Extension per ½ kg. Increase of Load.	Reading of Micrometer in mm.	Contraction per ½ kg. Decrease of Load.	Obsn. No.
1	0·0 *	0·256	—	0·260	0·160	21
2	0·5	0·418	0·162	0·420	0·160	20
3	1·0	0·577	0·159	0·580	0·160	19
4	1·5	0·738	0·161	0·740	0·156	18
5	2·0	0·896	0·158	0·896	0·159	17
6	2·5	1·055	0·159	1·055	0·160	16
7	3·0	1·214	0·159	1·215	0·157	15
8	3·5	1·370	0·156	1·372	0·156	14
9	4·0	1·528	0·158	1·528	0·157	13
10	4·5	1·685	0·157	1·685	0·155	12
11	5·0	1·840	0·155	1·840	—	11
Average extension per ½ kg. increase of load ....**0·1584**				Ditto for contraction **0·1584**		

Length of wire = 1 metre = 1000 mm.
Diameter d = 0·44 mm. = 0·044 cm.
Area of cross-section of wire = $\frac{1}{4}\pi d^2$ = 0·00152 cm.^{2}

Stress for a load of ½ kg. = ½/0·00152 = **328·9 (kgs. wt./cm.^{2}).**
Av. *strain* for same load = 0·1584/1000 = **0·0001584.**
Hence **Young's Modulus** Y = stress/strain = $3{\cdot}289 \times 10^6/1{\cdot}584$.
i.e Y = **2·08 × 10^6 (kgs. wt./cm.^{2}).**
or, converting the load to dynes, Y = **20·4 × 10^{11} (dynes/cm.^{2}).†**

---

* The weight of the hanger (½ kg.) is neglected, as merely serving to keep the wire taut.

† Kaye and Laby (*Physical and Chemical Constants*, 4th Ed.) give as the established range of values of Y for steel 19·5 to 20·6 × 10^{11} (dynes/cm.^{2}).

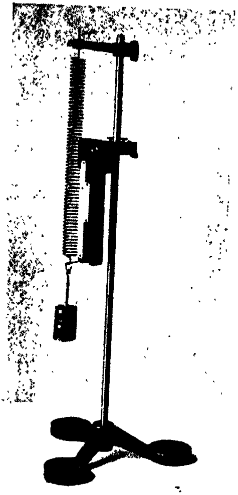


FIG. 19*e*.

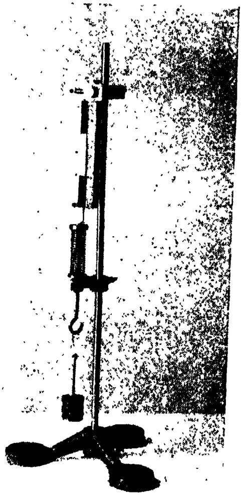


FIG. 19*f*.

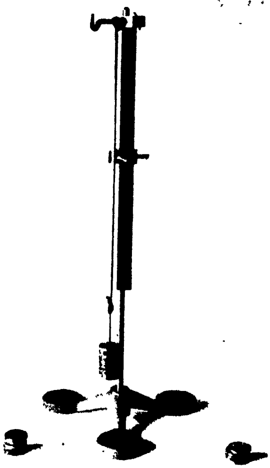


FIG. 19*g*.

Experimental Exercises.

(1) Verify, by means of such apparatus as is indicated in figs. 19*e*, 19*f*, that a spiral spring, whether under tension or compression, obeys Hooke's Law, if not unduly loaded.

Calculate also the *elastic modulus* * of the spring.

(2) By means of the apparatus shown in fig. 19*g*, verify that a rubber cord obeys Hooke's Law, if not unduly loaded.

Determine the *elastic modulus*,* and Young's Modulus for the material.

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* See Appendix II., Note 1.

* EXPERIMENT XX.

YIELD-POINT AND BREAKING-STRESS OF A WIRE.

Object of Experiment:

(1) **To show that if a wire be stretched by a gradually increasing load past the elastic limit, the ratio of strain to stress † no longer remains constant, but increases with increasing load;**

(2) **To measure the stress at the** *Yield-point,* † **and the** *breaking-stress.*

Apparatus:

A piece of thin wire of length about a metre (the material should be wrought iron or mild steel), special double vernier, scale and upper attachment (fig. 20), stout wooden beam supported about 6 ft. from the ground, kilo. (or lb.) hanger and slotted weights up to 20 kg. (or 40 lbs.), small coconut mat, wooden stool, micrometer screw gauge, 2 pieces of steel wire to support the double scale and hanging ring-weight.

Method:

(1) Set up the apparatus as shown in fig. 20.

It should be arranged that the loaded hanger does not fall more than a few inches when the wire is ruptured.

To this end place a stool with a small coconut mat on it underneath the hanger, leaving a gap of about 4 in. between (or the weight may be arranged to hang just clear of the floor; but the former is the most convenient arrangement).

(2) Measure the length of the wire from the top to the zero of the vernier.

(3) Take the reading of the vernier when the kilo hanger only is on the wire.

---

† See Appendix II., Note 1.

(4) Add weights in steps of 1 kg.—noting the vernier reading for each fresh load—until the neighbourhood of the elastic limit is reached; this may be ascertained roughly by noticing when the increase of extension per kg. is no longer constant.

(5) Continue adding kg. weights, keeping a look-out for the *Yield-point;* this is reached when a further addition to the load causes an abnormally large extension. (See Appendix II., Note 1.)

(6) Continue adding weights in *tenths* of a kg. until the wire breaks.

If proper precautions have been taken, the load should only have to fall an inch or two. The hanger should be caught with one hand, to prevent the weights being spilt.

(7) Measure the diameter of the wire in cm. at the point of rupture with the micrometer gauge—taking the mean of several readings in different directions, in case the wire has become oval.

Hence calculate the **Breaking-strain** of the wire; and the **Breaking-stress** (in kgs. wt. per sq. cm.).

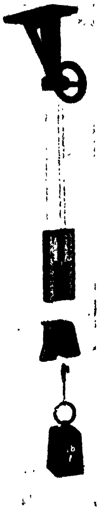

FIG. 20.

(8) All readings should be taken down *at the time* in tabular form; and a graph should afterwards be drawn exhibiting the way in which the extension varies with the load.

EXPERIMENT XXI.

Object of Experiment:

To find a means of neutralising the retarding friction of the wheels of a trolley, so that the only force acting on it parallel to the plane is the applied force.

Apparatus:

Fletcher trolley[*] *and plane* (figs. 21*a*, 21*b*); *wood blocks, ruler, spirit-level, special paper, thin sheet lead.*

Description of Apparatus.

This apparatus, by means of which a number of dynamical principles may be investigated, consists of :—

(1) *The trolley :* This is a rectangular block of wood about 20 ins. long (mass about 2 lbs.) mounted on three wheels, two on one side, and one on the other; the friction of the bearings being reduced as much as possible. The effective mass of the trolley is conveniently increased by drilling holes in the side of the wood block and inserting a number of metal cylinders, each of mass about ¼ lb. (See figs. 21*a*, 21*b*.)

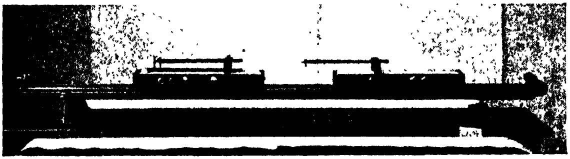

FIG. 21*a*.

(2) *The plane* on which the trolley runs may simply be a smooth wooden plank, accurately a plane surface. It is best,

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[*] The first trolley apparatus, of which the one shown in these and succeeding figures is a development, was devised in 1904 by W. C. Fletcher, Esq., H.M. Chief Inspector of Secondary Schools.

however, to have guides for the wheels, to prevent the trolley from falling off the sides; and the most convenient arrangement is to have a **V**-groove on one side of the plane, in which the two wheels "in tandem" run, while the other wheel runs in a flat-bottomed channel. Figs. 21*a*, 21*b* show a steel plane of this type, which is mounted near one end so that it can turn about a horizontal axis, while at the other end it rests upon a single levelling-screw. The angular deflections of the plane are measured by a degree-scale fitted with a vernier reading to minutes of arc.

At each end of the plane are a pair of rubber buffers; between which is a pulley-wheel, to be used when it is desired to set the trolley in motion by means of a falling weight.

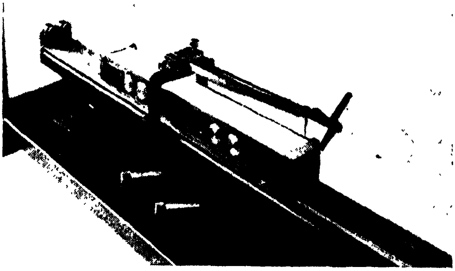

FIG. 21*b*.

(3) *The timing device:* Use is made of the fact that a flat steel spring fixed at one end will perform small oscillations which, though they are of decreasing amplitude, have a time of vibration, or period, which is very closely constant. (See Appendix II., Note 2*a*; also Ex. 4 on this Note.) The spring is mounted in a special holder —the "bridge-clamp"—which can be fixed to the side of the plane. This arrangement holds the spring so that it may perform horizontal vibrations above the trolley. An inked camel's-hair brush, fitted to the free end of the spring records on a strip of paper attached to the top of the trolley a "wave-tracing," which is an undulating line

arising from the combined motion of the spring and the trolley. (See, for instance, fig. 22*a*.) Knowing the period of the spring, the velocity of the trolley during this time may be deduced.

Most of the springs sold for this purpose have a period of $\frac{1}{10}$ sec. when clamped near the end, the correct length for this period being marked by the makers. In a number of experiments, however, it is unnecessary to know the period of the spring, for this period may be taken as the arbitrary unit of time.

The bridge-clamp shown in fig. 21*b* is also fitted with an "automatic release," a species of bell-crank lever, pivoted at the end of a horizontal rod. This lever holds the spring aside, and also holds the trolley in check when the plane is slanted. On turning the lever, the spring and the trolley are set in motion simultaneously.

First Method:

This consists in tilting the plane at such an angle that the wheel-friction of the trolley is exactly neutralised by the component down the plane of the weight of the trolley. The correct angle of slant is found by trial.

(1) First adjust the plane, by trials with a spirit level, so that it is level in all directions, both longitudinally and transversely. Test also whether the berch on which it rests is level.

(2) Slant the plane by placing a wood block under one end. The operation (1) should have secured that the plane is now level transversely.

(3) *To test the correctness of a trial slant*, attach a piece of the special paper to the trolley, and ink the brush with another brush.

Place the trolley on the plane *so that two wheels run in the* **V***-shaped groove.*

Adjust the brush so that it makes a fine line on the paper.

Without pulling the spring aside, draw the trolley through its own length and back, thus making a straight line down the middle of the paper.

(4) With the brush at the forward end of the trolley, pull the spring aside about *one quarter* of the width of the paper.

Give the trolley a push, at the same time releasing the spring.

When the correct slant has been reached, the wave-lengths so obtained will all be equal.

(5) The preliminary experiments may be made roughly, but on obtaining nearly the correct slant, careful measurements should be made; and, when the slant is as correct as you can make it, successive wave-lengths should be tabulated, as follows:—

TABLE XXI.

Measurement of Wave-tracing.

No of wave.	1	2, etc.
Wave-length ($= \lambda$)		
Diff. of successive wave-lengths ($= a$)		

(6) Tests should be made at different velocities, to see whether the slant is independent of the velocity.

(7) Measure the difference in height of the two ends of the plane, and its length. Hence calculate the gradient at which it departs from the horizontal.

Do not forget to test whether the bench is horizontal, and make allowances accordingly.

(8) A careful note should be made of this gradient, as it will be required in subsequent experiments.

This gradient will be referred to henceforward as the *friction-gradient.*

Second Method:

This consists in employing a "friction-rider" to neutralise the wheel-friction, the plane being horizontal.

The friction-rider consists of a weight—whose magnitude is exactly equal to the horizontal retarding force of the wheel friction—attached to the trolley by a thread passing over the pulley. It may be conveniently made from a piece of sheet lead; this admits of being easily cut down to the required weight.

(1) Carefully level the plane, both longitudinally and transversely, with the spirit-level. Place wood blocks under the centre to prevent it sagging when the trolley is on it.

(2) Attach a provisional rider to the trolley by a thread passing over the pulley at the end of the plane. Take wave-tracings, as in the first method, altering the weight of the rider until equal wave-lengths are obtained.

This rider should be preserved for use in subsequent experiments.

Exercises.

1. By measuring the weight of the trolley, and the angle of inclination of the plane to the horizontal when set at the friction-gradient, calculate the coefficient of friction between the trolley and the plane. (If the plane is not fitted with a degree-scale, the inclination may be found by measuring the difference in height between two points on the plane at a given distance apart.)

2. Check the above value of the coefficient of friction by finding the ratio of the weight of the friction-rider employed in the second method above to the weight of the trolley.

3. If the plane be tilted beyond the friction-gradient, so that h is the difference in height between two points distance x apart along the plane, show that if r' be the weight of the rider which must now be attached to the trolley in order to secure that the only force urging the latter down the plane is the resolved part of its weight in that direction, then

$$r' = r\sqrt{1 - h^2/x^2},$$

where r is the weight of the rider of Ex. 2.

Calculate the percentage decrease of r' from r when $x = 10h$.

(Ans.—0·5 $^{\circ}/_{\circ}$ approx.)

EXPERIMENT XXII.

Object of Experiment:

To show

(*a*) that a constant force produces a constant acceleration in a constant mass; and

(*b*) that the magnitude of the acceleration is directly proportional to the force.

Apparatus:

Trolley, plane, special paper, slotted weights, thread, wood blocks, spirit-level, ruler, large stand with adjustable platform.

SECTION I.

ELEMENTARY EXPERIMENT.

Method:

(1) *In order to establish* (*a*), first level the plane transversely by the method of the last Experiment; then tilt the plane up beyond the friction-gradient; e.g. put blocks about 5 in. high under one end.

(2) Take a wave-tracing in accordance with the instructions in EXPERIMENT XXI., using the automatic release shown in fig. 21*b*.

(3) *Caution:* Do not let the trolley bump against the lower end of the plane. The experiment only requires the trolley to run through its own length: it should therefore be caught with the hand when it has done so.

(4) Remove the paper and measure successive wave-lengths λ; hence find the acceleration a for each vibration. The mean acceleration for a vibration is the difference of the corresponding and the previous wave-length. The time of vibration of the spring may be taken as the unit of time. In a great many cases it is unnecessary to know the value of this unit in seconds.

Tabulate your results as follows :—

TABLE XXIIa.

Time (vibrations).	1	2, etc.
λ		
a		

(5) What is the force which is producing this acceleration?

(6) *In order to establish* (*b*), either the plane must be tilted at the friction-gradient, or the friction-rider (EXPERIMENT XXI.) must be used.

(7) Having carefully arranged that the wheel-friction is neutralised, attach 4/10 lbs. wt. to the trolley by a thread passing over the pulley. Arrange the adjustable platform so that it will receive this weight just before the trolley reaches the buffers at the end of the plane.

Using the automatic release, take a wave-tracing, bearing in mind the caution in (3).

(8) Similarly take wave-tracings with 6/10 and 8/10 lbs. wt. respectively as the applied forces.

Remove the paper and tabulate successive values of λ and *a* as below :—

TABLE XXIIb.

Acceleration is Proportional to Force (Mass Constant).

PART I.			PART II.			PART III.		
$F_1 = 0{\cdot}4$ lb. wt.			$F_2 = 0{\cdot}6$ lb. wt.			$F'_3 = 0{\cdot}8$ lb. wt.		
T	λ	*a*	*T*	λ	*a*	*T*	λ	*a*
1			1			1		
2, etc.			2, etc.			2, etc.		
Mean value of *a* $= a_1 = \dots$ $a_1/F_1 . = \dots$ °/o difference from mean = ...			Mean value of *a* $= a_2 = \dots$ $a_2/F_2 . = \dots$ °/o difference from mean = ...			Mean value of *a* $= a_3 = \dots$ $a_3/F_3 . = \dots$ °/o difference from mean = ...		
Mean value of a/F . = ...								

To assist the student in making the necessary calculations from his wave-tracing, a specimen wave-tracing is given in fig. 22*a*.

The successive wave-lengths AB, BC, CD, etc., of this tracing are carefully measured with a pair of compasses and a ruler. These are the values of λ in the table below.

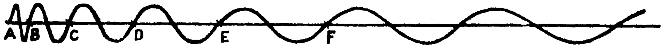

FIG. 22*a*.

From this wave-tracing, AB = 1·00 cm., BC = 1·40 cm., CD = 1·85 cm., DE = 2·30 cm., EF = 2·75 cm., etc.

The table below gives the complete calculations for two wave-tracings. Here the proportionality of the acceleration to the force is tested by computing the values of a_1/a_2 and F_1/F_2.

TABLE XXIIc.

Acceleration is Proportional to Force (Mass Constant).

Part I. Accelerating hanger 0·3 lb.			Part II. Accelerating hanger 0·6 lb.		
T	λ	a_1	T	λ	a_2
1	1·00	—	1	1·50	—
2	1·40	0·40	2	2·30	0·80
3	1·85	0·45	3	3·20	0·90
4	2·30	0·45	4	4·05	0·85
5	2·75	0·45	5	4·85	0·80
6	3·15	0·40	6	5·65	0·80
7	3·50	0·35	7	6·55	0·90
8	3·90	0·40	8	7·40	0·85
9	4·35	0·45	9	8·30	0·90
10	4·80	0·45			

Mean a_1 = 0·42; Mean a_2 = 0·85
Ratio of accelerating forces = 2·000
a_2/a_1 = 2·02
Error °/ₒ = 1·0.

SECTION II.

A More Accurate Experiment with a "Cord Tension Balance."

Note.—It has been assumed in Section I. that the force producing the acceleration a of the trolley is the whole weight of the accelerating hanger.

This, however, is not quite accurate.

If w gms.* be the mass of the hanger, N the number of vibrations per second of the spring, then, accepting provisionally the principle † which it is the object of EXPERIMENTS XXII and XXIII.—taken together—to establish, the force accelerating the trolley is actually

$$w(g - N^2a) \text{ dynes,}$$

instead of wg dynes,

i.e., the value of the accelerating force assumed in Section I. is too large in the ratio of

$$1 : 1 - N^2a/g.$$

Thus in the experiment detailed in Section I, Table XXIIb, where

$$a = 0{\cdot}422 \text{ (cm./vibn.}^2\text{)},$$

$$N^2a/g = 25 \times 0{\cdot}422/981 = 0{\cdot}0107;$$

so that the assumed value of the accelerating force is about 1 per cent. too large.

In the second part of the experiment, where $a = 0{\cdot}85$, the assumed value of the accelerating force is about 2 per cent. too large, introducing—as is seen above—an error of about 1 per cent. in the ratio of the two accelerating forces.

This error is eliminated by using a Cord Tension Balance (fig. 22*b*), which is simply a special form of spring-balance graduated in hundreths of a pound and reading from 0 to about 0·8 lb. The balance is screwed to the end of the trolley, and the thread supporting the accelerating hanger is attached to the lever of this balance.

The actual value of the force accelerating the trolley may by this means be read off on the scale of the balance during the motion.

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* As the mass of the hanger does not enter into the formulæ below, it is immaterial whether it is measured in lbs. or gms.

† i.e., that Force = mass × acceleration.

The experiment detailed in the first Section should be repeated, using the Cord Tension Balance.

The arrangement of this experiment is left as an exercise for the student.

Theoretical Exercises.

1. If N be the number of vibrations per second of the spring, a the acceleration in cm. per vibration per vibration, and b that in cm. per second per second, prove that

$$b = N^2a.$$

2. Establish theoretically the result referred to in Section II, viz., that if w gms. be the mass of the accelerating hanger, the force accelerating the trolley is

$$w(g - N^2a) \text{ dynes.}$$

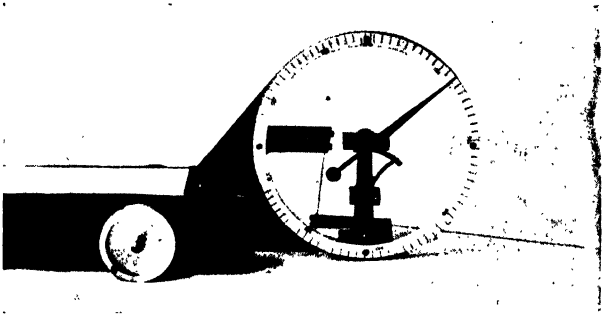

FIG. 22*b*.—Cord Tension Balance.

3. Prove that if in EXPERIMENT XXII., the mass of the accelerating hanger in Part II. were k times as large as that in Part I. then

$$a_2/a_1 = k.\{1 - N^2(a_2 - a_1)/g.\}$$

approximately.

State the degree of approximation.

Experimental Exercises.

1. It is an important principle of Theoretical Mechanics, based on experiments initiated by Galileo and others, that *the*

acceleration produced in a body by gravitational attraction is independent of the mass of the body.

This principle, as far as it applies to motion on a plane inclined to the vertical, may be investigated with the apparatus of EXPERIMENT XXII. as follows :—

(i) Using the trolley with all its loading-cylinders in, tilt the plane well beyond the friction-gradient, and, after providing for the neutralisation of wheel-friction, take a wave-tracing from rest and so determine the acceleration of the trolley.

(ii) Reduce the mass of the trolley by removing one of the loading-cylinders. Without altering the slant of the plane, repeat (i).

(iii) Proceed in this way to measure the acceleration of the trolley for as many different values of the trolley-mass as possible; thus, if the trolley be fitted with six loading-cylinders, seven different values of the trolley-mass would be available.

(iv) This series of observations may be repeated for a different slant of the plane.

The only experimental difficulty lies in providing for the neutralisation of the wheel-friction. It is to be observed that the coefficient of friction between the trolley and the plane will be different for each observation, since it depends on the pressure on the wheel-bearings, and, therefore, on the weight of the trolley. It will, therefore, be most convenient first to make a set of friction-riders * for each value of the trolley-mass, by the second method of EXPERIMENT XXI., and at the outset of each observation to attach the appropriate rider to the trolley.

2. A method alternative to the use of a Cord Tension Balance for measuring the force actually accelerating the trolley in the above Experiment, i.e. another method of correcting for the inertia of the falling hanger, is as follows :—

Tilt the plane well beyond the friction-gradient, and attach to the trolley a thread passing over the pulley at the top of the plane; to the thread attach a weight

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* In making these riders, the point noticed in Ex. 3, p. 80, should not be forgotten.

(*w* lbs. wt.) whose size is to be adjusted until the trolley, when given a push, will move down the plane with constant velocity. *w* is then a measure of the effective force which, if it were not so neutralised, would give the trolley an acceleration *a* down the plane.

The value of *a* may be determined by removing *w*, and allowing the trolley to roll from rest down the plane.

EXPERIMENT XXIII.

Object of Experiment:

To show that the acceleration produced by a constant force is inversely proportional to the mass acted upon.

Apparatus:

Trolley and plane, special paper, spirit-level, hanger, and slotted weights, thread, friction-rider, stout balance capable of weighing the trolley, large stand with adjustable platform, wood blocks.

Method:

The method consists in measuring the acceleration produced in the trolley by the same force in seven cases; in each case the mass of the trolley is increased by the insertion of a metal cylinder fitting into its body (see fig. 21*b*, p. 77).

(1) Adjust the plane so as to be accurately horizontal both longitudinally and transversely, the pulley projecting over the edge of the bench.

(2) Remove the 6 metal cylinders in the body of the trolley, and weigh the trolley on the balance, correct to ·01 lb.

Also weigh each of the cylinders; these should be numbered 1 to 6.

(3) Attach the friction-rider and take a preliminary wave-tracing, to make sure that the friction-rider is accurate.

(4) Add 4/10 lbs. to the thread, and take a wave-tracing, using the automatic release. The platform should be adjusted at such a height that it receives the hanger after the trolley has moved a little more than its own length.

(5) Tabulate successive values of λ and a, and hence obtain the mean acceleration a_1 for the experiment.

Also form the product a_1M_1, where M_1 is the mass of the trolley.

(6) Perform 6 other similar experiments, in each case increasing the mass of the trolley by the addition of a metal cylinder, the load on the thread remaining unaltered.

It is the object of the experiment to test whether the product aM is the same for each case.

Tabulate your results as follows :—

TABLE XXIII.

a varies as $1/M$ (F const.)

Part I.			Part II.			Part III.
M_1 = ... lbs.			M_2 = ... lbs.			etc., for remaining 5 cases.
T	λ	a	T	λ	a	
1			1			
2, etc.			2, etc.			
Mean value of a $= a_1 = \ldots$ $a_1M_1 = \ldots$ °/o diff. from mean = ...			Mean value of a $= a_2 = \ldots$ $a_2M_2 = \ldots$ °/o diff. from mean = ...			
Mean value of aM = ...						

Note.—The magnitude of the retarding force due to the friction of the wheels in their bearings depends upon the pressure on these bearings, and therefore upon the weight of the trolley.

A different friction-rider, therefore, will be required in each of the seven cases; or, if the friction-gradient method be used, a different gradient will be required in each case. (See p. 86.)

EXPERIMENT XXIV.

Object of Experiment:

(*a*) **To measure the velocity of the trolley** *at a given instant*, **when acted on by a constant force; and hence to establish the relation** $v = at$,

(*b*) **to establish the relation** $U = \frac{1}{2}(v_1 + v_2)$, **where** v_1 **and** v_2 **are the velocities at the beginning and end of a given interval of time, and** U **is the average velocity * over that interval.**

Apparatus:

Trolley and plane, special paper, spirit-level, hanger and slotted weights, thread, friction-rider, Capstan stand with adjustable platform,† wood blocks.

Method:

It should be clearly realised that the measurement of a wave-length gives the *average velocity over that interval* (in centimetres per vibration); actually the velocity is continually changing throughout the vibration.

The following is a method of measuring the actual velocity at any assigned instant of the motion.

(1) *To establish* (*a*), neutralise the wheel-friction by either of the methods of EXPERIMENT XXI.

(2) After the necessary adjustment of the plane, take a wave-tracing, using the automatic release.

(3) Adjust the platform so that it receives the weight exactly at the end of the 5th vibration. Call this the point A (fig. 24*b*).

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* See Appendix I., § 11.

† A convenient platform is made from two Capstan blocks placed end to end, and united by a short $\frac{3}{8}$ in. steel pin fitted into the pivot holes.

(4) Now bring the trolley back to the zero position (the point O), and take another tracing on the same paper.

The brush should retrace its path up to the end of the 5th wave-length, and thereafter the wave-lengths should all be equal. Any one of these equal wave-lengths gives the velocity at the end of the 5th vibration.

(5) As, in practice, these wave-lengths are rarely exactly equal, it is better to measure the total length of 3 or 4 of them, and take the mean of these as the value of v_1, the velocity at the end of the 5th wave-length (in cm. per vibration).

Measure also the distance OA.

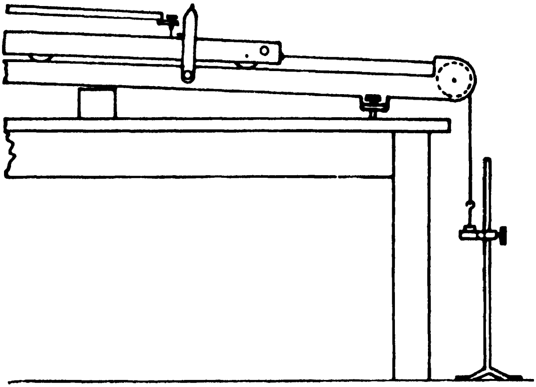

FIG. 24a.

From the wave-tracing obtain the mean acceleration a_1 for the first 5 vibrations.

Compare the above value of v_1 with that of a_1t_1, where t_1 is the time from the start measured in vibrations.

Express the difference between v_1 and a_1t_1 as a percentage of v_1.

(6) Put a fresh paper on the trolley, and take a complete wave-tracing under the same conditions as before.

(7) Adjust the platform so that it receives the weight exactly at the end of the 10th vibration. Call this the point B (fig. 24c).

(8) Now bring the trolley back to the zero position—the point O—and take another tracing on the same paper.

The brush should retrace its path up to the end of the 10th wave-length, and thereafter the wave-lengths should all be equal. Any one of these equal wave-lengths gives the velocity at the end of the 10th vibration.

(The arrangement of apparatus is shown in fig. 24*a*.)

(9) As, in practice, these wave-lengths are rarely exactly equal, it is better to measure the total length of 3 or 4 of them, and take the mean of these as the value of v_2, the velocity at the end of the 10th vibration.

Measure also the distance OB. Hence obtain the length AB. From the wave-tracing obtain the mean acceleration a_2 for the first 10 vibrations.

Compare the above value of v_2 with that of a_2t_2, where t_2 is the time from the start, measured in vibrations.

Express the difference between v_2 and a_2t_2 as a percentage of v_2. This and (5) give two experimental tests of the formula

$$v = at.$$

The following, which is the result of an actual experiment, shows the way in which the results should be tabulated :—

TABLE XXIV.

Part I. Test of $U = at$.

t_1	a_1	a_1t_1	v_1	Error °/₀	t_2	a_2	a_2t_2	v_2	Error °/₀
5	·675	3·375	3·4	0·74	10	·631	6·31	6·35	0·63

Part II. Test of $V = \frac{1}{2}(v_1 + v_2)$.

v_1	v_2	AB	$t_2 - t_1$	$U = \text{AB}/(t_2 - t_1)$	$\frac{1}{2}(v_1 + v_2)$	Error °/₀
3·4	6·35	24·35	5	4·87	4·875	0·10

To assist the student in making the necessary calculations from his wave-tracings, two specimen wave-tracings are given in figs. 24*b* and 24*c*.

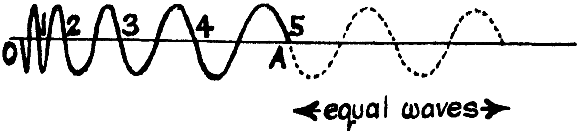

FIG. 24*b*.—Wave-tracing showing accelerating hanger removed at the end of the fifth wave-length. (The numbers indicate the ends of the corresponding waves.)

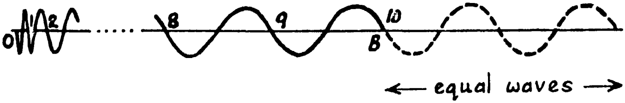

FIG. 24*c*.—Wave-tracing showing accelerating hanger removed at the end of the tenth wave length. (In order to economise space, the waves from the 3rd to the 7th are omitted.)

The procedure is the same as in EXPERIMENT XXII.; a_1 and a_2 are the mean acceleration for the first and second wave-tracings respectively.

EXPERIMENT XXV.

Object of Experiment:

To verify the Principle of the Conservation of (linear) Momentum, by observation of the impact of two trolleys.

Apparatus:

Two Fletcher trolleys, steel or wooden plane, wave-tracing spring and brush, and bridge-clamp for the same, roll of wave-tracing paper, camel's-hair brush to ink the wave-tracing brush.

For some of the Exercises a second wave-tracing spring, brush, and bridge-clamp are required. Exercise (3) *also requires a spiral spring.*

A convenient method of giving an impulse to the bombarding trolley is by means of a flat spring, e.g. another wave-tracing spring. The arrangement of this apparatus is shown in fig. 28, *EXPERIMENT XXVIII.*, *p.* 109. *The details are—* 2 *Capstan stands,* 3 *Capstan blocks,* $\frac{3}{8}$ *in. steel cross-rod held to Capstan stands by* 2 *retort-stand clamps, G-clamp opening to about* 3 *inches.*

A number of experiments in illustration of the Principle of the Conservation of linear momentum may be performed with this apparatus.

Experimental details are given below for the simplest of these, while others are suggested to the student in the Exercises on pp. 97-99.

Details of the Simplest Experiment.

The experiment (fig. 25*a*) consists in the impact of one trolley with a second of equal mass. The bombarding trolley is fitted at the front with a needle point projecting about an inch. This drives into a cork fitted in the rear of the stationary trolley; so that after the impact the two proceed with a common velocity, which, if the masses of the two trolleys are exactly equal, should be half the bombarding velocity.

The necessary impulse is conveniently given to the bombarding trolley by the spring device outlined in fig. 25*a*, and more fully described and illustrated in EXPERIMENT XXVIII. The advantage of this method of giving the impulse is, that it is possible to reproduce a given velocity of impact, with fair accuracy, any number of times; while, by altering the degree of thrust of the spring, a wide range of bombarding velocities may be obtained.

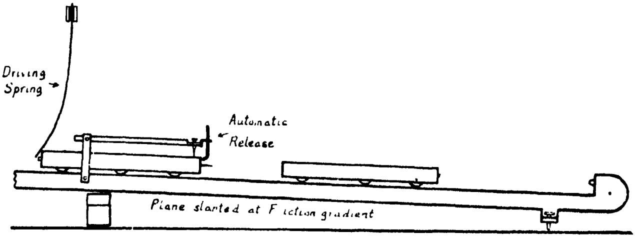

FIG. 25*a*. — Bombardment by moving trolley (left) of stationary trolley (right). (For simplicity the supports of the driving-spring and automatic release have been omitted.)

Let M_1, M_2 = masses of bombarding and stationary trolleys, respectively;
V_1 = velocity of bombarding trolley immediately before impact (in inches/vibration).
V_2 = common velocity of the two trolleys immediately after impact (same units).

Then it is the object of the experiment to show that

$$M_1 V_1 = (M_1 + M_2).V_2.$$

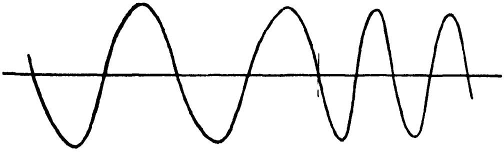

FIG. 25*b*.

Details of an Actual Experiment.

M_1 = 6·03 lbs.; M_2 = 6·00 lbs.;
V_1 = 1·92 inches/vibration;
V_2 = 0·97 „

The frequency of the spring (value not actually required) was 10 per second.

Hence $M_1 V_1 = \mathbf{11{\cdot}58}$ units of momentum;

$(M_1 + M_2) \, . \, V_2 = \mathbf{11{\cdot}67}$ „ „

Mean of these $= 11{\cdot}625$;

Deviation of either

from mean = **0·39 per cent.**

The wave-tracing from which these results were obtained is illustrated in fig. 25*b*, above.

Experimental Difficulties.

The student should find this a fairly simple experiment to carry out; the following necessary adjustments may, however, be pointed out:

(1) It is important that the wheel-friction of the trolleys should be neutralised; and it is most convenient to do this by tilting the plane at the friction-gradient. Now the friction-gradient may not be exactly the same for both trolleys; it is, however, immediately after impact that it is most necessary for the wheel-friction to be absent.

Preliminary experiments should, therefore, be made with the two trolleys jammed together with the needle point in order to ascertain this gradient. If a second wave-tracing spring and bridge-clamp are available, it is convenient to make the wave-tracings for ascertaining the friction-gradient on the forward trolley (M_2), though this is not necessary.

(2) The use of another wave-tracing spring to give an impulse to the bombarding trolley has the advantage of making the whole operation automatic. The method of arranging this will readily be grasped from fig. 28, EXPERIMENT XXVIII. The bombarding trolley, with the end of this driving spring pressing against it, is held stationary by the projecting stop of the automatic release (see fig. 21*b*, EXPERIMENT XXI.); by depressing the lever of this release, the whole experiment takes place automatically.

(3) Before starting a collision, a zero-line should be drawn on the wave-paper attached to the trolley. This is, of course, done by drawing the trolley under the wave-brush when the latter is stationary.

The instant at which the collision occurs is indicated by bringing the bombarding trolley into contact with the stationary one, and while in this position making a cross-mark with the wave-brush on the zero-line. Care must be taken, in disengaging the two trolleys, that the one which is to be bombarded is not displaced.

(4) The student may experience some difficulty in determining the velocities immediately before and after impact, owing to the circumstance that the wave-tracing may not cross the zero-line exactly at the place where the cross-mark, previously made to indicate impact, is situated.

In order to get over this difficulty, he should proceed as follows: Let I (fig. 25*c*) be the position of impact (cross-mark). Through I draw a straight line at right angles to the zero-line, cutting the wave-tracing in the point O.

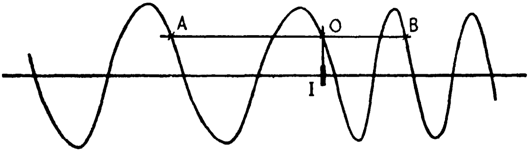

Fig. 25*c*.

Through O draw a straight line parallel to the zero-line, cutting the wave-tracing previously and subsequent to O in the points A, B. Then OA, OB are the distances described by the bombarding trolley in unit time immediately before and after the impact, and hence may be taken as the measures of V_1 and V_2 respectively.

Exercises.

1. Carry out experiments similar to the one outlined above (i.e. with trolleys of equal mass), using as wide a range of bombarding velocities as possible. Different velocities are obtained by altering the thrust of the driving spring on the back of the trolley.

2. Perform other experiments with trolleys of unequal mass. If each trolley is fitted with four removable metal cylinders of mass 1 lb., five different values may be

obtained for the mass of each trolley. (What is the total number of different experiments thus obtainable?)

It should not be forgotten that any alteration of mass of either trolley involves a re-adjustment of the friction-gradient.

3. Another extremely instructive set of experiments may be performed as follows:

Make both trolleys of equal mass, and slant the plane at the friction-gradient. Place the trolley M_1 behind M_2 (i.e. higher up the plane) with the needle-point at the back.

To the front of M_1 fasten a spiral spring, which is to be connected to the back of M_2 by a piece of thread about 6 inches long. Starting with the thread slack and M_1 at rest, give M_2 a push down the plane, so that, when the thread becomes taut, M_1 is also set in motion.

Observe carefully the resulting motion. What conclusions do you draw from this experiment?

Wave-tracings must be taken for the motion of each trolley. This necessitates two wave-tracing springs, brushes, and bridge-clamps.

The experiment may be varied by altering the masses of the trolleys.

(For further information on this set of experiments, see Ashford, *Elementary Dynamics*, p. 163.)

4. Placing the two trolleys so that the ends fitted with ivory studs are opposed to one another, make experiments to test the truth of *Newton's Law of Impact* for the direct collision of two elastic bodies:

Let u, v be the velocities of the bombarding trolley immediately before and after impact, and v' that of the bombarded trolley immediately after impact; then Newton's Law asserts that

$$\left\{\begin{array}{l}\text{Relative velocity of}\\ \text{the two bodies im-}\\ \text{mediately after im-}\\ \text{pact}\end{array}\right\} = -e \left\{\begin{array}{l}\text{their relative velo-}\\ \text{city immediately}\\ \text{before impact}\end{array}\right\}$$

which, in this case, gives that

$$v - v' = -e.u,$$

the quantity e being known as the *coefficient of restitution* for the two impinging bodies.

The student should, then, set himself to ascertain, by means of as wide a range of bombarding velocities as possible whether the ratio

$$(v' - v)/u$$

is, within the limits of experimental error, a constant quantity.

EXPERIMENT XXVI.

Change of Kinetic Energy = work done.

Apparatus:

Trolley on plane slanted at the friction-gradient, small slotted weights and hanger, Capstan stand fitted with adjustable platform, strong balance capable of weighing the trolley, spirit-level, ruler.

Method:

Outline.—The mass of the trolley is first found by weighing it on a strong balance (a delicate chemical balance should not be used), with slotted weights in the opposite pan.

The trolley is then placed on the plane, which has been slanted at the friction-gradient, and an accelerating hanger attached to it by a thread passing over the pulley at the end of the plane.

A wave-tracing is then taken from rest, and the velocity

$$v \text{ (cm. per vibration)}$$

after x cm. of the motion determined by the method of EXPERIMENT XXIV.

Theoretical:

Let M gms. = mass of the trolley,
w ,, = mass of the accelerating hanger,
N = no. of vibrations per second of the spring.

(It will be seen below that only the ratio of M to w is required, hence it is immaterial whether gm. or lb. weights are used; metric units are employed in this Theoretical treatment on account of their superior convenience.)

The work done by the accelerating hanger in falling the distance x cm. is

$$wgx \text{ ergs.}$$

This is expended mainly in giving Kinetic Energy to the trolley, the final value of which is

$$\tfrac{1}{2}M \,.\, N^2v^2 \text{ ergs.}$$

Kinetic Energy is also given to the falling hanger, the final value of which is

$$\tfrac{1}{2}w \,.\, N^2v^2 \text{ ergs.}$$

Since the Kinetic Energy of the system is initially zero, the total *change of Kinetic Energy* is

$$\tfrac{1}{2}(M + w) \,.\, N^2v^2 \text{ ergs.}$$

Thus it is the object of the experiment to test whether

$$\tfrac{1}{2}(M + w) \,.\, N^2v^2 = wgx,$$

i.e. whether

$$\tfrac{1}{2}(M/w \,.\, + 1) \,.\, N^2v^2 = gx, \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

a form more convenient for the purposes of calculation.

Procedure:

The student who has successfully performed the previous Experiments should have no difficulty in performing the present one without any further directions; it may be pointed out, however, that it will be found convenient to adjust the platform so that the distance x is a whole number of wave-lengths.

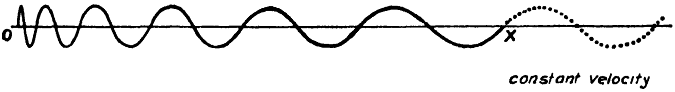

FIG. 26.

A typical wave-tracing is given here (fig. 26), together with the numerical calculations of the corresponding experiment.

In this figure OX = x = 27·6 cm. = distance moved with accelerated velocity. After X the waves are of equal length, giving velocity at X = v = 9·7 cm. per vibration.

Details of Specimen Experiment:

M = 7·47 lbs.,*
w = 0·5 ,,

---

* The wheel-friction was neutralised by the second method of EXPERIMENT XXI., the plane being horizontal. The mass of the friction-rider was 0·13 lb.; this is included, for convenience, in the value of M. It is to be observed that, while the work done by the falling rider equalises that done against friction, by virtue of its velocity the rider increases the total Kinetic Energy of the system. This is represented by an addition of 0·13 lb. to the moving masses.

N = 6 vibrations per second,
x = 27·9 cm.,
v = 9·7 cm. per vibration,
g = 981·3 cm. per second per second.

Hence M/w. = 14·94, $1 + M/w$. = 15·94,

v^2 = 94·09, and N^2v^2 = 36 × 94·09.

Thus L.H.S. of (1) = $\frac{1}{2}$ × 15·94 × 36 × 94·09
= **27,000**

R.H.S. of (1) = gx = 981·3 × 27·9
= **27,380**

The average of these = 27,190
and the per cent. difference
of either from the average = ± 0·7.*

---

* That is, the computed value of the Kinetic Energy is about $1\frac{1}{2}$ per cent. less than that of the work done. The reason of this comparatively large error is mainly that the Kinetic Energy of rotation of the three wheels of the trolley, and the pulley-wheel at the end of the plane, have been neglected. An approximate correction for this is obtained by adding to the total mass of the moving system *half the mass of each wheel.* (See p. 120 ff.; also Ex. 7, p. 128.)

The student who performs any trolley experiment for a second time should consider how this correction for the "inertia" of the wheels is likely to affect his results.

* EXPERIMENT XXVII.

THE MOTION OF A WHEEL-AND-AXLE ON AN INCLINED PLANE.

Object of Experiment:

To verify the principle

Change of Kinetic Energy = Work done……… (1)

for the motion from rest of a solid wheel rolling on its axle through a measured distance down an inclined plane.

Apparatus:

Solid wheel and axle, both of the same metal; plane with slot cut in it to admit of the rolling of the wheel, wave-tracing spring and brush, bridge-clamp with horizontal slot to allow the spring to vibrate in a vertical plane; some means of measuring the angle of inclination of the plane to the horizontal.†

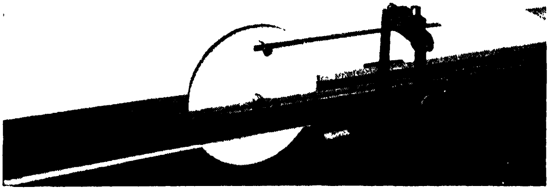

Fig. 27*a*.—The motion of a wheel-and-axle on an inclined plane.

The apparatus is illustrated in fig. 27*a*.

---

† In the experiment detailed below, the steel plane shown in fig. 28 was used. This plane was fitted with an inclination-scale reading to minutes of arc.

Outline of Method:

Since the wheel starts from rest, the change of kinetic energy during the motion will be equal to its final value; hence the final value of the velocity of the wheel must be measured. This is obtained directly by causing the spring to record a wave-tracing on the side of the wheel (which is painted white to facilitate this record), for the space of a few inches on either side of the final position. A representation of the type of wave-tracing obtained is given in fig. 27*b*.

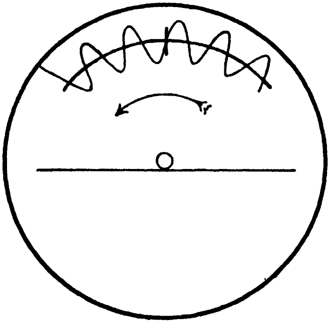

FIG. 27*b*.—Type of wave-tracing recorded on the wheel.

It is this method of measuring the final velocity of the wheel which is the outstanding feature of the experiment.

A method of measuring the exact distance rolled by the wheel, as well as of recording the velocity at the exact position chosen for the finish, is discussed under the heading "Experimental Difficulties" below.

Measurements Required.

a = radius of wheel (ft.),
b = ,, ,, axle (ft.),
L = distance (ft.) which wheel-and-axle rolls down plane,
θ = angle of inclination of plane to the horizontal,
V = terminal velocity (ft./sec.) of centroid (axis) of wheel-and-axle,
U = terminal velocity (ft./sec.) recorded by the wave-spring and brush on the side of the wheel (the frequency of the wave-tracing spring must be known),
d = depth (ft.) below rim of wheel of zero-line of this wave-tracing.
c = radius of zero-line = $a + b - d$.

Theoretical Treatment.

Calculation of Change of Kinetic Energy (= K ft.-poundals).

Let ω = terminal angular velocity (radians/sec.) of wheel-and-axle,

M lbs. = mass of wheel-and-axle (value not required),
I = its moment of inertia about its centroid (axis)
= $\frac{1}{2}Ma^2$ (see EXPERIMENT XXXVII.; also Appendix II., Note 6). neglecting $M.I.$ of axle.

Then we have

$$K = \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}MV^2,$$
$$= \tfrac{1}{4}Ma^2U^2/c^2. + \tfrac{1}{2}Mb^2U^2/c^2. \quad (\text{since } \omega = U/c., \text{ and } V = bU/c.),$$
$$= MU^2(a^2 + 2b^2)/4c^2 \quad \text{ft.-poundals.}$$

The *work done* during this motion is

$$W = MgL\sin\theta \quad \text{ft.-poundals.}$$

The relation (1) then gives

$$MU^2(a^2 + 2b^2)/4c^2. = MgL\sin\theta.$$

We have thus to verify that

$$U^2(a^2 + 2b^2)/4c^2. = gL\sin\theta,$$

or that

$$K/M. = W/M.$$

Results of an Experiment.

a = 4 in. = 0·3333 ft.,
b = $\frac{3}{16}$ in. = 0·01563 ft.,
d = 0·72 in. = 0·06 ft.,
L = 21·8 in. = 1·816 ft.,
U = 39·75 in./sec. = 3·313 ft./sec.,
θ = 3° 33′.

This gives $K/M.$ = **3·674,**
and $W/M.$ = **3·620,**

Mean = 3·647,
difference = 0·027
= **0·74 per cent.**

Experimental Difficulties.

The main difficulties in the way of securing good results in this experiment are included under the following headings:—

(1) Before setting the wheel into motion, it must be carefully adjusted so that its axis is accurately at right angles to the line of greatest slope of the plane; otherwise it will catch against the side of the slot in which it runs before coming to the end of its journey.

This adjustment may be secured by inserting two flat pieces of metal of exactly the same thickness—e.g. two $\frac{1}{16}$-lb. slotted weights—at the side of the wheel, and at opposite ends of a diameter. Press these against the side of the slot with the wheel, and then carefully remove them without disturbing the setting of the wheel.

(2) In order to hold the wheel stationary at the beginning of its journey, while making other adjustments, a short bar magnet may be placed in front. This will be found to give sufficient hindrance, and it is easily removed without disturbing the setting of the wheel.

(3) The brush of the wave-tracing device must be made to record three things on the side of the wheel:—

(i) **The zero-line of the wave-tracing:** Obtained by rolling the wheel with the brush stationary.

(ii) **The end of the journey:** Obtained by making a cross-mark on the zero-line when the axis of the wheel is in the position chosen for the finish. The wheel may be held in this position by means of a second bar magnet.

(iii) **Several complete waves on either side of the final position (and not more than this):** The beginner may experience a difficulty here owing to the fact that the zero-line recorded by the stationary brush is a spiral, one convolution of which corresponds with a comparatively short distance of roll down the plane (a little over $\frac{1}{2}$ in. in the case detailed above).

To overcome this difficulty, it will be best to carry out the operations (i) and (ii) at the outset of the experiment and then to roll the wheel up the plane—taking care that it does not slip—until it has reached the position chosen for the start. It is at this moment that the adjustment described in (1) should be applied.

To obtain only the few waves required, the brush should be held aside (upwards) with the hand, and released when the axis of the wheel is about $\frac{1}{4}$ in. from the final position.

Exercises.

1. Assuming that the wheel and the axle are made of the same material, calculate the percentage alteration in the value of K obtained by taking into account the Moment of Inertia of the axle.

2. If the necessary apparatus is available, the following is an instructive experiment on the relation between the *translational* and *rotational* Kinetic Energies of a wheel rolling on its axle.

A wheel (fig. 27c), whose axle is built in two stages of different diameters, is set rolling upon the portions AA of its axle down a pair of rails slanted at the friction-gradient. After it has rolled some distance, it is made to continue its journey on the portions BB of the axle over a second pair of rails so arranged that, while there is a sudden change from the one pair to the other, the geometrical axis of the wheel continues in the same straight line. The result is a marked increase in angular velocity and decrease in translational velocity.

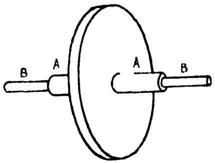

FIG. 27c.

The explanation lies in the fact that, since the moving system possesses no Potential Energy, the sum of the translational and rotational Kinetic Energies must be constant.

The correctness of this assertion may be tested by taking wave-tracings on the side of the wheel—after the method of EXPERIMENT XXVII.—for the two different portions of its journey. (Take the two wave-tracings on opposite sides of the wheel, and so calculate the two Kinetic Energies in each case.)

The development of this experiment is left as an exercise to the student.

*EXPERIMENT XXVIII.

Object of Experiment:

To verify the Principle of the Conservation of Momentum by observation of the impact of a trolley with a loaded beam, free to swing about a vertical axis. (Fig. 28.)

Apparatus:

Steel plane fitted with levelling-screw and carrying at one end a projecting bracket supporting a vertical pivot on ball-bearings. (On this pivot swings the "Impact-arm," a rectangular block of wood with the pivot near its centre. The mass of this impact-arm is increased by the addition of removable metal cylinders, symmetrically placed with regard to the pivot.) One trolley, fitted with ivory stud in front; 2 wave-tracing springs, brushes, and bridge-clamps (one of the latter carrying an automatic release), roll of wave-tracing paper, camel's hair brush to ink the wave-tracing brushes.

A desirable addition is also a device for giving an impulse to the trolley, consisting of:—Flat steel spring (a large wave-tracing spring), 2 Capstan stands, 3 Capstan blocks, $\frac{3}{8}$-in. steel cross-rod held to Capstan stands by 2 retort-stand clamps, G-clamp opening to about 3 inches.

The complete apparatus is shown in fig. 28.

Description of Experiment.

This apparatus affords another method of studying the Principle of the Conservation of Momentum; it has also the merit of giving the more advanced student a practical acquaintance with angular momentum. By assuming the mathematical developments involved, however, the less advanced student may still perform the experiment without troubling himself about angular momentum.

The experiment consists in causing the trolley to bombard the impact-arm. Wave-tracings are taken which record the velocities of the trolley before and after impact, and the velocity of recoil of the impact-arm, which is at rest before the collision.

The trolley is fitted in front with an ivory stud which impinges upon a similar stud on the impact-arm. The latter carries a special platform to which the wave-tracing paper is attached.

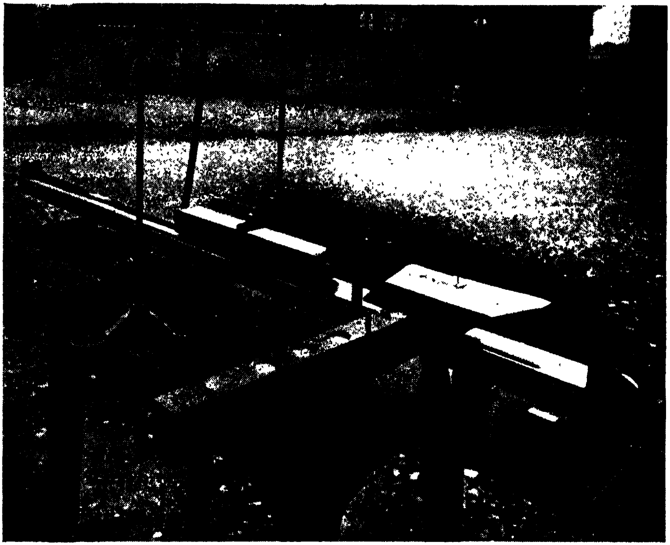

FIG. 28.

Theoretical Discussion.

It may be shown that, as far as the question of momentum is concerned, the motion of the impact-arm immediately after the collision is equivalent to the linear motion along the line of impact of a single mass, J lbs.* J is, in fact, the mass of an "equivalent trolley."

---

* For a proof of this, see Appendix II., Note 9.

Let W = mass of bombarding trolley (in lbs.);
while J = mass of "equivalent trolley" to impact-arm.
Also, let U_1, U_2 = velocities of bombarding trolley immediately before and after the collision, respectively;
and V = velocity of impact-arm immediately after the collision.
(All these velocities are measured in inches/vibration.)

Then the total momentum of the system immediately before impact is

$$WU_1 \quad \text{(units of momentum)},$$

while that immediately after impact is

$$WU_2 + JV$$

in the same direction.

It is thus the object of the experiment to show that, within the limits of experimental error,

$$WU_1 = WU_2 + JV,$$

or that

$$W(U_1 - U_2) = JV.$$

A number of different values of U_1, and therefore also of U_2 and V, may be obtained by altering the thrust of the driving spring. In all these cases, however, provided that the masses of the bombarding trolley, the impact-arm, and its loading weights, are kept constant, we shall have

$$\mathbf{(U_1 - U_2)/V. = J/W. = \text{a constant quantity.}}$$

It is, then, suggested that the student should carry out experiments to ascertain whether the left-hand member of the above equation is, within the limits of experimental error, an invariable quantity.

TABLE XXVIII.

Details of a Specimen Experiment.

Obsn. No.	U_1	U_2.	V.	$U_1 - U_2$.	$(U_1 - U_2)/V$
1	1·00	0·60	0·72	0·40	**0·556**
2	1·30	0·78	0·93	0·52	**0·559**
3	1·61	0·98	1·13	0·63	**0·558**
4	1·70	1·03	1·20	0·67	**0·558**
5	1·95	1·19	1·36	0·76	**0·559**
6	2·13	1·32	1·45	0·81	**0·559**
7	2·50	1·55	1·70	0·95	**0·559**

This set of observations, then, establishes the constancy of the quantity

$$(U_1 - U_2)/V$$

with a marginal variation of 3/559,

or **0·55 per cent.**

between the extremes.

The frequencies of the two springs need not be known; but it is, of course, important that they should both be the same.

(In the above experiments the frequency of each spring was, as a matter of fact, 10 per second.)

Experimental Difficulties.

Apart from the adjustment of the spring device for giving impulse to the trolley, the experiment should not present much difficulty to the student.

The following points may be noted :-–

(1) The plane must be slanted at the friction-gradient. (The apparatus used in the experiment above-described was so constructed that the axis of the Impact-arm remained vertical at whatever angle the plane was slanted.)

(2) Before the wave-tracings are taken, zero-lines should be drawn with the wave-brushes on the wave-papers which are fastened to the trolley and the impact-arm.

Then, setting the impact-arm carefully at right angles to the length of the plane, bring the trolley up so that its ivory stud is just in contact with the ivory stud of the impact-arm.

While the two are in this position, make cross-marks with the respective brushes to indicate the instant of impact. Care must be taken that the impact-arm is not, before the collision, displaced from the position corresponding to the cross-mark.

(3) The wave-tracing spring for the trolley should be set in motion by the automatic release, which at the same time releases the driving spring. The other wave-tracing spring may be started by hand, a second automatic release being hardly feasible, as it is liable to interfere with the subsequent motion of the impact-arm.

(4) If, after the experiment, it is found that the wave-tracing for the trolley has not crossed the zero-line at the cross-mark (the instant of impact), the student should adopt the plan suggested in EXPERIMENT XXV. for a similar difficulty. (See p. 97.)

Experimental Exercises.

It is suggested that the more advanced student should, after reading Notes 8 and 9 of Appendix II., make use of the above apparatus to determine the Moment of Inertia :—

(1) Of a rectangular block of wood about an axis through its centroid.

(2) Of a metal cylinder about an axis parallel to its own axis (hence determine the M.I. about its own axis).

(3) Of a body of irregular shape about any given axis.

The arrangement of these experiments is left to the initiative of the student.

EXPERIMENT XXIX.

ATWOOD'S MACHINE.

SECTION I.

Apparatus:

Atwood's machine attached to a vertical frame, so that the wheel is about 4 ft. from the floor; the whole must be capable of being levelled so that the rim of the wheel is horizontal transversely; special paper; spirit-level.

SECTION II.

Description of Instrument.

The instrument consists, essentially, of a wheel Y (fig. 29*a*), made as light as possible, and mounted in nearly frictionless ball-bearings.

On this wheel is hung a band of paper of length about 4 feet, to the ends of which are attached 2 equal masses W, W′ (0·75 lb. each.)

In order that there may always be equal weights of paper on either side of the wheel, another band of the same paper also of length about 4 feet is attached to the lower ends of W and W′. (For the special method of attaching the paper to the weights, see Procedure (2) below.)

P is a platform, hinged at H, and supported by the catch K. A suitable device provides for the automatic dropping of the platform at the appropriate instant, as explained below.

A variety of experiments may be performed with this instrument, and in all of them the following procedure is adopted:—

W is brought to rest on the platform P, and upon it is placed a U-shaped rider of mass m (the instrument is provided with several such riders of given masses).

Below P, at a point whose position may be varied, is clamped to the rod R a U-shaped catch-plate D which is constructed so that it allows W to pass unhindered, but catches the mass m.

If now the platform be dropped, W and m fall through the distance between P and D with accelerated velocity, W′ rising with equal acceleration.

On reaching D, the rider m is removed, and, the equality of the masses on either side having been restored, the system continues with constant velocity, namely that acquired when m reaches D.

Above the wheel is fixed a flat steel spring S, carrying an inked brush at its end.

By this means a wave-tracing is made on the paper band, from which the velocity and acceleration at any instant of the motion may be deduced.

The catch K supporting the platform P is attached to a vertical rod Q, to which is also attached the catch C which holds the spring S aside.

On turning the lever L through a small angle, K is moved aside, and at the same time C is made to ascend. Thus the platform is dropped and the spring set in motion simultaneously.

Atwood's machine is primarily designed to measure the value of g, but one and the same wave-tracing may also be employed to establish several other simple Dynamical principles.

Directions are given below for using the instrument to investigate the following principles:—

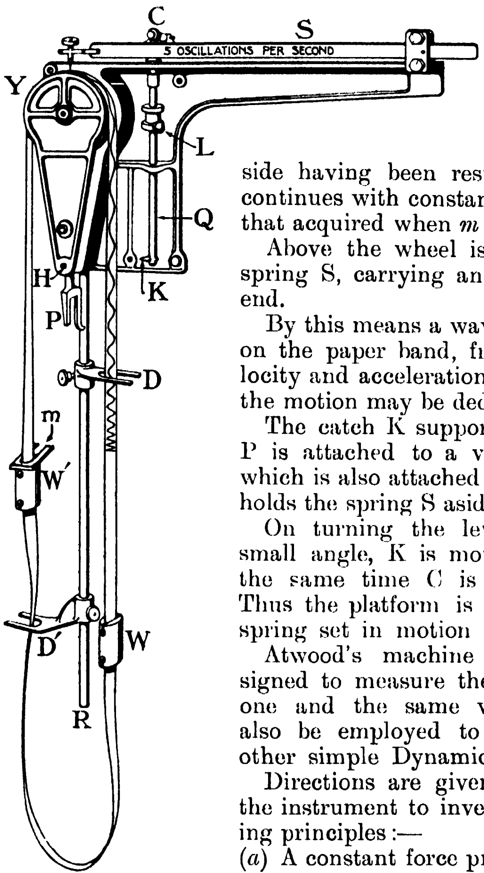

FIG. 29a.

(a) A constant force produces a constant acceleration in a constant mass. (An alternative to Experiment XXII.)

(b) $v = at$ (An alternative to Experiment XXIVa.)

(c) $s = \frac{1}{2}at^2$.

(d) $U = \frac{1}{2}(v_1 + v_2)$. (An alternative to Experiment XXIVb.)

(*e*) Determination of the value g.
(*f*) Verification of the Principle of the Conservation of linear Momentum.

SECTION III.

Procedure:

(1) Level the apparatus with a spirit-level so that the rim of the wheel is horizontal transversely.

(2) Cut off a piece of the special paper about 4 feet long, and attach its ends to the masses W, W′ in the following manner:—

Double an end of the paper over twice to a depth of about half an inch, and press the folds flat (fig. 29*b*).

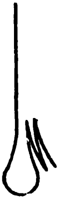

FIG. 29*b*.

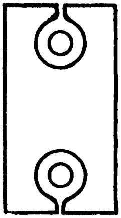

FIG. 29*c*.

Now bend the end of the paper so folded into a loop about an inch deep, slip the loop sideways on to the pin at the top of W, and pull up the paper until the folded part jams in the narrow slit (fig. 29*c*).

The paper is now held securely.

Care should be taken that the paper hangs vertically downwards when W and W′ are vertical.

The other end of the paper should be similarly attached to W′.

(3) Cut off another piece of the same paper about 4 feet long, and attach it to the lower ends of W and W′ by the method of (2).

The whole thus forms a continuous loop, and ensures that there are always equal weights of paper on either side of the wheel.

(4) Carefully lift the paper band on to the wheel, and see that it lies evenly on either side.

(5) Fix the catch-plate D about 1 foot below P (the actual distance moved under accelerated velocity will be recorded on the wave-tracing).

The catch-plate D′ should be fixed about half-way up the rod R so as to act as a guide for the paper on the other side.*

(6) With the platform P hanging down, test by moving the system up and down that D and D′ will allow the weights W and W′ to pass through them unhindered.

(7) *To test whether the wheel has been correctly levelled* :—

Ink the brush on the end of the spring S with another brush, and adjust it so that a fine line is made on the paper.

Now, keeping the brush motionless in the centre of the paper, raise W to the top and let it slowly down to its furthest position.

The brush should draw a straight line down the centre of the paper; and, moreover, the paper should still be lying evenly on either side of the rim of the wheel.

If, however, the paper winds off the wheel during this process, the wheel has not been correctly levelled.

In this case the wheel must be re-levelled, and, if necessary, a fresh paper band used.

(8) *To eliminate the retarding action of the wheel-friction* :—

Place a friction-rider on top of W.† Now raise W to its highest position.

Set the spring vibrating with one hand, and, simultaneously, with the other hand give W a push downwards.

If the friction-rider is accurate, the wave tracing thus obtained should have all its wave lengths equal.

Test this by removing the paper and measuring successive wave lengths. (An approximate estimate of the accuracy may be obtained by testing the wave-lengths with a pair of compasses while the band is still on the wheel.)

---

* Fig. 29*a* gives a general view of the instrument, but does not show it completely adjusted for this particular experiment; for our purposes D should be lower down, and *m* should be on W, and not on W′.

† A convenient friction-rider for this purpose may be made from a piece of No. 20 copper wire, bent into a U; the correct size being found by the method of EXPERIMENT XXI.

If the friction-rider is inaccurate, it should be adjusted, by clipping if too heavy, and by the addition of a small spot of solder if too light.

(9) Having carefully tested and arranged that the wheel friction is neutralised, the weights should be fitted to a new paper band.

The machine is now ready to investigate the principles (*a*) to (*e*).

With the exception of (*d*) all the results (*a*) to (*e*) may be deduced from one and the same wave-tracing.

(*d*) requires that a second wave-tracing be taken with a different distance between P and D: (*f*) requires a separate tracing.

(10) Lift the new band on to the wheel, and test as in (8) that it does not run off as W moves down.

Move W up above the platform, and raise the latter, fixing the catch K underneath it.

The brush is still to remain stationary at the centre of the paper.

(11) Place the friction-rider on W, and let the latter down on to the platform. (Care should be taken to place the friction-rider so that it will not be interfered with by D.)

At the side of the brush make a mark on the paper to indicate the starting-point. Call this point A.

(12) Now place one of the U-shaped masses *m* on top of W in such a position that it will be caught later on by the catch-plate D.

Holding W with one hand, move the lever L aside with the other, thus dropping the platform.

Now carefully let the system down until *m* is just caught by D.

Still holding the band or the weight with one hand, make a mark on the paper at the side of the brush (the point B). This marks the end of the accelerated motion.

(13) Raise W again, and, holding it with one hand, with the other lift the platform and fix the catch K under it. At

the same time pull the spring S aside and slip the catch C over it so that it is held.

Let W down on to the platform.

The apparatus is now finally ready for the taking of the wave tracing which is to test the principles in question.

(14) Pull the lever L aside, thus setting the system in motion.

The part AB is traversed with accelerated velocity.

After the point B the velocity should remain constantly at the magnitude acquired at B.

The motion should be stopped before W′ quite reaches the wheel, otherwise the paper will be torn.

The paper should now be detached from the weights.

SECTION IV.

Deduction of Results from the Wave-Tracing.

In order to test :—

(*a*) Measure successive wave-lengths carefully with a ruler. Tabulate these, and their differences, as in EXPERIMENT XXII., Table XXIIc.

(*b*) Find the average acceleration a_1 for the whole motion, from the above table.

Measure the total length of several wave-lengths after the point B (fig. 29*d*), divide by the number of these, and hence obtain the mean value of v_1, the velocity at the point B (in cm. per vibration).

Compare the value of v_1 with that of a_1t_1, where t_1 is the time from A to B measured in vibrations.

Express the excess (+) or defect (−) of a_1t_1 over v_1 as percentage of v_1.

(*c*) Measure the distance AB (= s_1 c.n.), and form the product

$$\tfrac{1}{2}a_1t_1^2.$$

Calculate the excess (+) or defect (−) of this over s_1 as a percentage of s_1.

(*d*) To establish this, another wave-tracing should be taken on a fresh paper with the same mass *m*, but with AB (now A′B′) a longer distance, i.e. with D below its previous position.

As in (*b*), by measuring the total length of several wave-lengths after the point B′, obtain v_2, the mean value of the velocity at the point B′ (in cm. per vibration).

Measure the distance A′B′ ($= s_2$ cm.).

Hence obtain U, the average velocity between B and B′ viz.:—

$$(s_2 - s_1)/(t_2 - t_1),$$

where t_2 is the time from A′ to B′ measured in vibrations.

Calculate the value of $\frac{1}{2}(v_1 + v_2)$, and express the excess (+) or defect (−) of this over U as a percentage of U.

By forming a similar table for this second wave-tracing, the value of a_2, the mean acceleration from A′ to B′ may be found.

Hence an additional experimental test of (*b*) and (*c*) is provided.

(*e*) Determination of the value of g:–

(1°) *Disregarding the inertia of the wheel Y.*

Let T (dynes) = tension in the paper band during the motion,

a = acceleration in cm. / (vibration)^2,

b = acceleration in cm. / sec.^{2},

N = number of vibrations per second of the spring S.

Then by Newton's Second Law of Motion we have for the downward motion of W

$$(W + m) \,.\, g - T = (W + m) \,.\, b \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

and for the upward motion of W′

$$T - W \,.\, g = W \,.\, b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

By the addition of (i) and (ii)

$$mg = (m + 2W) \,.\, b, \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

or

$$g = (1 + 2W/m.)b \ldots\ldots\ldots \ldots\ldots \text{(iv)}$$

But

$$b = N^2 a.$$

Hence

$$\boldsymbol{g = (1 + 2W/m) \,.\, N^2 a} \ldots\ldots\ldots\ldots \text{(v)}$$

The quantities W and m are known, and the value of N is marked on the spring.

Hence a numerical value of g is given

(2°) *Correction for the inertia of the wheel Y.*

It might be supposed, that, having provided for the neutralisation of the friction of the wheel bearings by a friction-rider, all the potential energy given up by the falling mass m would be employed in giving kinetic energy to the moving masses m, W and W′

This, however, is not quite true, as an appreciable fraction (about 6 per cent. for this particular instrument—see Section IV. below) of the potential energy is wasted in giving kinetic energy to the wheel.

The actual acceleration b of the system is therefore less than that which would give the correct value of g from equation (iii).

It will now be shown that equation (iii) must be emended by the addition to the masses on the right hand side of a quantity y which may be termed the "**equivalent mass**" of the wheel.

The emended form of equation (iii) is then

$$mg = (m + 2W + y) \,.\, b, \dots\dots\dots\dots\dots \text{(iii}a\text{)}$$

where y remains to be determined. (It is shown below that y is approximately half the mass of the wheel.)

Let Y = mass of wheel,
R = radius of wheel,
k = radius of gyration of the wheel about its axis (see Appendix I., § 19),
V = velocity of the moving masses (in cm./sec.) at any point of the motion.

Then the kinetic energy acquired by the wheel is (see EXPERIMENT XXXVII.),

$$\tfrac{1}{2} Yk^2V^2/R^2,$$

and that of the masses m, W and W′ is

$$\tfrac{1}{2}(m + 2W) \,.\, V^2.$$

Hence the total kinetic energy of the system is

$$\tfrac{1}{2}(m + 2W + Yk^2/R^2) \,.\, V^2,$$

so that the wheel converts into kinetic energy as much potential energy as would, in virtue of its motion, a mass

$$y = Yk^2/R^2 \dots\dots\dots\dots\dots\dots\dots\dots \text{(vi)}$$

This quantity is referred to as the "equivalent mass" of the wheel.

The numerical value of k^2/R^2, and therefore of y, may be found experimentally by the method of EXPERIMENT XXXVII. This is left as an exercise to the student.

A fairly good approximation to its correct value may be obtained by regarding the wheel as a uniform cylindrical disc of radius R.

In this case it is proved experimentally in EXPERIMENT XXXVII. that

$$k^2/R^2 = \tfrac{1}{2},$$

hence $$y = \tfrac{1}{2}Y, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(vii)}$$

i.e. the equivalent mass is approximately *half the mass of the wheel.*

With the help of (iii*a*) and (vi) we therefore have as the emended value of g :—

$$g = (1 + 2W/m\,. + Y/2m)\,.\,N^2a \ldots\ldots\ldots\ldots\ldots \text{(va)}$$

(*f*) **Verification of the Principle of the Conservation of Linear Momentum**:—

This experiment requires a fresh paper band, and employs a retarding rider (mass m') placed on the left-hand catch-plate D′.

This rider is picked off the catch-plate D′ by the ascending weight W′; an impact thus occurs causing a decrease in the velocity of the system.

The positions of the two catch-plates should be arranged by trial so that the retarding rider m' may bring the system to rest before W reaches its lowest position.

Now the Principle of the Conservation of Momentum asserts that

Total Momentum before Impact = Total Momentum after Impact.

Let v_1 = velocity (in cm./sec.) just before the rider is picked up,
v_2 = velocity (in cm./sec.) just after the rider is picked up,
m' = mass of retarding rider (in gms.),
W = mass of each of the suspended weights (in gms.),
y = equivalent mass of the wheel (in gms.).

The total momentum just before impact is

$$(2W + y)\,.\,v_1, \qquad \text{(gm.-cm./sec.)}$$

that just after impact is

$$(2W + y + m')\,.\,v_2, \text{(gm.-cm./sec.)}.$$

Thus it is the object of the experiment to test whether

$$(2W + y)\,.\,v_1 = (2W + y + m')\,.\,v_2 \ldots\ldots\ldots \text{(viii)}$$

Let B be the point of the wave-tracing at which the accelerating hanger m is removed, and C the point at which m' is caught up; then, since the portion BC is described with constant velocity v_1, v_1 may be determined by measuring the total length ($= x$ cm.) of a whole number of wave-lengths between B and C, and dividing by the time T in seconds.

Thus $$v_1 = x/T.$$

v_2 may be found by measuring the distance ($= z$ cm.) from the point C to the end E of the wave-tracing.

If a' be the retardation for this part of the motion, which is found by measuring successive wave-lengths of the portion CE, then v_2 is given by the formula

$$v_2^2 = 2a'z.$$

The numerical details of an experimental test of this Principle will be found under Section V. below.

To estimate the degree of accuracy of the experiment

($a°$) Calculate the numerical value of each side of equation (viii), and the mean of the two

($b°$) Express the difference between either side of (viii) and the mean as a percentage (+ or −) of the latter.

SECTION V.

Experimental Results.

The two wave-tracings were taken under the same conditions, the only difference between the two being that the second was longer than the first. For convenience only 12 wave-lengths of each are recorded. A representation of wave-tracing No. 1 is given in fig. 29d.

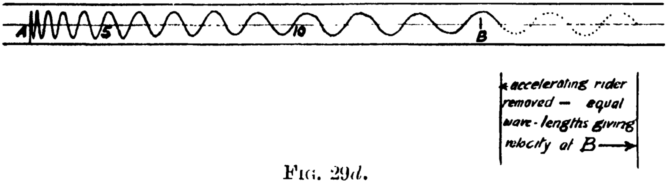
A
5
10
B
accelerating rider removed — equal wave-lengths giving velocity of B

FIG. 29d.

Test of (a), viz.:—A constant force produces a constant acceleration in a constant mass.

The constant force is the weight of the accelerating rider.

TABLE XXIX.

Details of the First Two Wave-Tracings.

Tracing No. 1.			Tracing No. 2.		
t (vibrations.)	λ (cm.)	a (cm./vibn.^{2})	t (vibrations.)	λ (cm.)	a (cm./vibn.^{2})
1.........	0·54	0·51	1.........	0·51	0·51
2.......	1·03	0·49	2.........	1·02	0·51
3.........	1·49	0·46	3.........	1·50	0·48
4.........	1·97	0·48	4...	1·98	0·48
5.........	2·46	0·49	5.........	2·46	0·48
6.........	2·95	0·49	6.........	2·96	0·50
7.........	3·44	0·49	7.........	3·43	0·47
8.........	3·95	0·51	8.........	3·90	0·47
9.........	4·45	0·50	9.......	4·37	0·47
10.........	4·93	0·48	10.........	4·85	0·48
11.........	5·40	0·47	11.........	5·36	0·51
12.........	5·85	0·45	12.........	5·84	0·48
Mean value of a $a_1 = 0·4875$. Total time from A to B $t_1 = 13·4$ vibns. Distance from A to B $s_1 = 43·06$ cm. Accelerating rider $m = 0·02$ lb. $W = 0·75$ lb. Constant velocity after the removal of the rider $v_1 = 6·48$ cm./vibn.			Mean value of a $a_2 = 0·4867$. Total time from A to B $t_1 = 15·9$ vibns. Distance from A to B $s_2 = 61·10$ cm. Accelerating rider $m = 0·02$ lb. $W = 0·75$ lb. Constant velocity after the removal of the rider $v_2 = 7·84$ cm./vibn.		

It will be seen from Table XXIX. that, though the acceleration fluctuates from one vibration to another, the mean value for the first tracing, viz.:—

$$0·4875 \text{ cm./vibn.}^2$$

is sensibly equal to that for the second, viz.:—

$$0·4867 \text{ cm./vibn.}^2;$$

the mean of these two is

$$0·4871 \text{ cm./vibn.}^2,$$

and the deviation of either from this mean is 0·08 per cent.

We are justified in attributing the fluctuations referred to above to the short-comings of the instrument.

Thus, within the limits of experimental error, we may regard Principle (*a*) as established.

Test of (*b*), viz.:—$v = at.$:

Tracing No. 1 gives

$$v_1 = 6{\cdot}48 \text{ cm./vibn.}$$
$$a_1t_1 = 6{\cdot}53 \quad ,,$$

and the per cent. difference of the latter from v_1

$$= +\,0{\cdot}77$$

Tracing No. 2 gives

$$v_2 = 7{\cdot}71 \text{ cm./vibn.}$$
$$a_2t_2 = 7{\cdot}74 \quad ,,$$

and the per cent. difference of the latter from v_2

$$= +\,0{\cdot}39$$

Giving an average error of + 0·58 per cent.

Test of (*c*), viz.:—$s = \frac{1}{2}at^2$:

Tracing No. 1 gives

$$\tfrac{1}{2}a_1t_1^2 = a_1t_1 \times \tfrac{1}{2}t_1$$
$$= 43{\cdot}75$$
$$s_1 = 43{\cdot}06$$

and the difference of the last two as a percentage of s_1

$$= +\,1{\cdot}60$$

Tracing No. 2 gives

$$\tfrac{1}{2}a_2t_2^2 = a_2t_2 \times \tfrac{1}{2}t_2$$
$$= 61{\cdot}53$$
$$s_2 = 61{\cdot}10$$

and the difference of the last two as a percentage of s_2

$$= +\,0{\cdot}70$$

Giving an average error of + 1·15 per cent.

Test of (*d*), viz.:—$U = \frac{1}{2}(v_1 + v_2)$:

From the above we have

$$U = (s_2 - s_1)/(t_2 - t_1). = 18{\cdot}04/2{\cdot}5 = 7{\cdot}22$$
$$\tfrac{1}{2}(v_1 + v_2) = \tfrac{1}{2} \times 14{\cdot}19 = 7{\cdot}10$$

per cent. difference of the latter from $U = -\,1{\cdot}67$.

Test of (*e*), viz.:—$g = (1 + 2W/m. + Y/2m.).\,N^2a$......... (v*a*)
Taking for the value of a the mean of a_1 and a_2 as given by Table XXIX. we have from the above

$$\begin{aligned} a &= 0{\cdot}4871 \text{ cm./vibn.}^2, \\ W &= 0{\cdot}75 \text{ lb.}, \\ m &= 0{\cdot}02 \text{ lb.}, \\ Y &= 0{\cdot}20 \text{ lb.}, \\ N &= 5 \text{ vibrations per second}, \\ \text{hence } g &= (1 + 75 + 5) \times 25 \times 0{\cdot}4871 \\ &= 81 \times 12{\cdot}1775 \\ &= 986{\cdot}378 \text{ cm./sec}^2. \end{aligned}$$

Now the correct value for the Latitude of the Laboratory is (see Appendix I., § 20):—

$$g = 981{\cdot}281 \text{ cm./sec.}^2$$

Thus the value as given by Atwood's Machine differs from the correct value by

+ 0·52 per cent.

Test of (f), viz.:—**The Principle of the Conservation of Momentum:—**

The experiment detailed below was performed with another similar instrument. The quantities observed were:

Mass of each suspended body	$W = 375$ gms.,
Equivalent Mass of wheel ...	$y = 42$ gms.,
Mass of retarding rider.........	$m' = 10$ gms.,
Measured distance travelled with uniform velocity before impact	$x = 13{\cdot}56$ cm.,
Time thus occupied	$T = 2$ vibrations $= 0{\cdot}4$ sec.
Distance moved between impact and rest..................	$z = 46{\cdot}78$ cm.,
Mean acceleration between impact and rest	$a' = 11{\cdot}84$ cm./sec^2.

So that the velocity just before impact .... $v_1 = x/T = 13{\cdot}56/0{\cdot}4$
$= \mathbf{33{\cdot}9}$ cm./sec.,

and the velocity v_2 just after impact is given by the equation ... $v_2^2 = 2a'z = 2 \times 11{\cdot}84 \times 46{\cdot}78$
$= 1108$

thus $v_2 = \mathbf{33{\cdot}29}$ cm./sec.

Hence the **Momentum of the system just before impact** $= (2W + y) . v_1$
$= \mathbf{26{,}849}$ gm.cm./sec.,

and the **Momentum of the system just after impact** $= (2W + y + m') . v_2$
$= \mathbf{26{,}699}$ gm.cm./sec.

The mean of these $= 26{,}774$ gm.cm./sec.,
so that the deviation of either from the mean ……………… $= 0{\cdot}28$ per cent.

SECTION VI.

List of Symbols Employed.

W = Mass of each weight supporting the paper band (all masses are in lbs. unless otherwise stated).
m, m' = Masses of accelerating and retarding riders.
Y = Mass of wheel.
y = Equivalent Mass of wheel ($= \frac{1}{2} Y$ approximately).
R = Radius of wheel.
k = Radius of Gyration of wheel about its axis.
a = Acceleration of the system in cm./vibn.
b = ,, ,, ,, in cm./sec.
v = Velocity after time t. (In cm./vibn. unless otherwise stated.)
U = Average velocity (in cm./vibn.).
N = Frequency of spring S (in vibrations per second).
g = Acceleration of gravity in cm./sec^2.

Exercises on Experiment XXIX.

1. Account for the fluctuations from one vibration to another of the value of a in Table XXIX.

2. Calculate the value of g for the above experiment neglecting the correction for the Equivalent Mass of the wheel. (See eqn. (v.) above.)

   Estimate the percentage error introduced by neglecting this factor.

3. Determine experimentally the value of the Radius of Gyration of the machine by the method of EXPERIMENT XXXVII.

Hence find the Equivalent Mass of the wheel, and the percentage difference of this value from the value $\frac{1}{2}Y$ assumed in Section IV. (*e*).

4. The wheel of the Atwood's Machine employed in EXPERIMENT XXIX. is not, as there assumed, a uniform disc; it approximates more nearly to a uniform disc surmounted by a circular cylinder of the same material and thickness.
   State whether you expect the true value of the Equivalent Mass to be less or greater than $\frac{1}{2}Y$.
   Give a discussion of this question without using any Mathematical formulæ.

5. Obtain a theoretical expression for
   (i) the Radius of Gyration,
   (ii) the Equivalent Mass,

   of the wheel about its axis by supposing it to consist of a uniform disc of radius R surmounted by a circular cylinder, symmetrically placed, of width L; both being composed of the same material of thickness θ.
   Calculate from your formula the numerical value of these two quantities in the following case (that of the instrument used in Section IV. (*e*)):

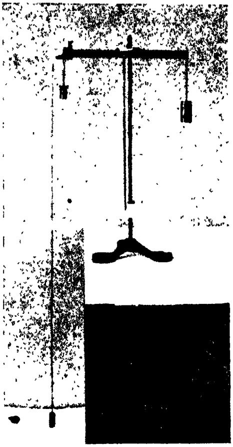

Fig. 29*e*.

$$Y = 0{\cdot}2 \text{ lb.}, R = 6{\cdot}325 \text{ cm.}, L = 2{\cdot}5 \text{ cm.}, \theta = 0{\cdot}13 \text{ cm.}$$

Compare these with the corresponding quantities obtained experimentally in Exercise 3.

6. Set up apparatus similar to that shown in fig. 29*e*, and perform the following experiment:—
   Place 2 equal masses W on the ends of the string passing over the pulley on the left; and adjust the size of the counterpoise K on the right until the lever is in horizontal equilibrium.
   Now redistribute the masses on the pulley string so

that one is $W - m$, and the other $W + m$. When the system is left to itself, what effect does this distribution have on the equilibrium of the lever? How do you account for this effect, and what general dynamical principle does it illustrate?

(It is most convenient to have m small compared with W, so that the system moves slowly at first.)

Calculate the tension in the string for both the above distributions of the masses.

(Ans. Wg; $g(W^2 - m^2)/W$.)

7. The three wheels of the trolley used in EXPERIMENT XXVI. were each of mass 0·04 lb., while the mass of the pulley-wheel at the end of the plane was 0·08 lb.

Calculate the total "equivalent" mass of these four wheels, and hence re-calculate the left side of equation (1), p. 101, making this allowance for the inertia of the wheels.

Calculate also the percentage difference between this revised value for the kinetic energy, and that for the work done.

(Ans. 0·1 lb.; 27,330; 0·18 per cent.)

EXPERIMENT XXX.

SIMPLE PENDULUM.

Object of Experiment:

(*a*) **To show that the length L of a simple pendulum is proportional to the square of its time of vibration T;**

(*b*) **To find the value of g, the acceleration of a body falling freely to earth under gravity.**

Apparatus:

Lead sphere attached to a thread 110 *cm. long, the other end of the thread to be hung from a metal grip attached to a large stand; stop-watch; parallel-jaw calipers; metre rule.*

Method:

(1) Measure the diameter of the lead sphere with the calipers. As it may not be perfectly spherical, take readings of the diameter in 5 different directions, and take the mean of these as the correct diameter.

(2) Adjust the length of the thread so that the total length from the grip to the centre of the lead sphere ($= L$) is 100 cm.

(3) Pull the sphere aside about 2 in., and let it go, taking care that the sphere swings in a vertical plane, and does not describe a loop.

(4) Take the time of 20 complete vibrations with a stop-watch. Hence find the time ($= T$) of one complete vibration. This should be done three times, and the mean taken.

The time-measurement should be taken from the centre of the swing (why?). A complete vibration is thus the time that elapses from the instant when the pendulum passes the centre—from left to right, say—until the instant when it again passes the centre in the *same direction.*

The timing should be done as follows:—

At the instant of the first passage through the centre from left to right, start the stop-watch, at the same time saying to yourself "Nought"; at subsequent passages say "one," "two," etc., up to "twenty."

In order to estimate accurately the instant of passage through the centre, some clearly visible mark should be set up behind the bottom of the pendulum when it is still at rest, e.g. a dark vertical line—fairly broad—on a piece of white paper.

The observer should also stand some distance—say 4 feet—from the pendulum in such a position that he sees the line ruled on the paper exactly behind the string of the pendulum, when the latter hangs freely at rest.

(5) Altering L to 90 cm., find the new time of vibration, taking the same number of observations as in (4).

TABLE XXX.

Experimental Results.

Number of Experiment	1	2, etc., to	8
Length L	100	90, etc., to	30
Time of vibration T			
T^2			

(6) Take similar observations for $L = 80, 70, 60$, down to 30 cm.

(7) Tabulate your results as in Table XXX. The values of T^2 may be found from a table of squares, or by a slide-rule.

(8) Plot a graph of L against T^2. What kind of graph do you expect?

(9) **Calculation of the Value of g:**

It may be shown theoretically * that g is given by the relation

$$T = 2\pi \sqrt{L/g},$$

hence $$g = 4\pi^2(L/T^2) \ldots\ldots\ldots\ldots \quad \text{(i)}$$

Find from your graph the best value of L/T^2. Hence, by substitution in (i), calculate the value of g.

(10) Compare your value so obtained with the correct value given by Helmert's Formula †, expressing the excess (+), or defect (−) of your value over the correct value as a percentage of the latter.

Experimental Exercises Suggested by Experiment XXX.

(The student should read Note 4, Appendix II., before doing these experiments.)

1. Suspend a spiral spring so that it hangs vertically downwards; then

   (i) Attach a $\frac{1}{2}$ lb. weight to the lower end, and when it has come to rest under this load, pull it down a further 4 inches, and then release it; thus starting a vertical oscillation of the weight.

   Determine the period of this oscillation and calculate the length of the simple equivalent pendulum (or, alternatively, determine it experimentally).

   (ii) Repeat operation (i) with weights of $\frac{3}{4}$, 1, and $1\frac{1}{4}$ lbs. respectively.

   (iii) Plot a graph showing how the period of oscillation depends on the magnitude of the weight attached to the spring.

   (The spring should not be too stiff, nor should it be so weak that the total extension demanded here strains it beyond the **elastic limit.** ‡

   A suitable spring may be quickly made by taking a piece of steel piano-wire of diameter about $\frac{1}{40}$ of an inch, and closely winding about 120 turns round a metal rod of diameter $\frac{1}{4}$ in. This is most conveniently done by fixing the rod in a lathe. If a lathe is not available, a substitute may be improvised with a bench drill.

---

* See Appendix II., Note 3.

† See Appendix I., § 20.

‡ See Appendix II., Note 1.

An assistant is required to keep the uncoiled portion tight during the winding. On releasing the spiral its diameter will be found to increase by about half, owing to partial uncoiling. It should be strained lengthways until successive turns are about $\frac{1}{10}$ inch apart.)

2. Perform a similar experiment with a rubber cord.

* EXPERIMENT XXXI.

Object of Experiment:

To show that the *Mean-Position-Velocity* of a simple pendulum varies directly as the amplitude of the arc of swing.

Apparatus:

Wooden drawing-board suspended by 2 steel wires meeting at a hook in the ceiling (in the experiment described below the distance from the hook to the board was 10 *ft.).*

(In order to secure an approximation to a simple pendulum, pairs of 1-kg. slotted weights were bolted to the board (fig. 31*a). This has the further advantage of eliminating torsional vibrations, and giving a steady swing.)*

A flat steel spring carrying a brush, such as is used in EXPERIMENT XXI.; special wave paper; drawing pins.

Fig. 31*a shows the essentials of the apparatus, while fig.* 31*b is a photograph of the actual apparatus.*

Outline of Method:

A piece of paper (e.g. the special wave paper, as employed in EXPERIMENT XXI.) is pinned along the centre of the board, and the spring S (fig. 31*a*) is fixed by means of clamps so that the brush is just in contact with the paper, and is at the centre of the paper when the pendulum is at rest.

Before taking a wave-tracing, the brush should be made to draw a line (actually an arc of a circle whose centre is the hook in the ceiling) along the paper, by moving the pendulum back and forth. The position of the brush when the pendulum is at rest—the point O (fig. 31*a*)—is then marked on the paper.

It is difficult to set the pendulum swinging truly in a vertical plane if it be started by hand: it should therefore be drawn aside by the thread EF, and set in motion by burning this thread.

In order to hold the pendulum aside, it is convenient to pass the thread over a pulley (fig. 31*b*), and attach sufficient weights to it to hold the pendulum aside.

A mark must be made on the paper indicating the position of the brush for this zero position of the pendulum—the point X (fig. 31*a*).

The spring is to be held aside, and released at the instant when the thread EF is severed.

The amplitude of the arc of swing is OX.

The *mean-position-velocity* is that at O.

In the experiment detailed below the mean of the two wave-lengths on either side of O was taken as the mean-position-velocity (measured in cm./vibn.).

In this way it is shown (see Table XXXI. below) that the fraction

(mean-position-velocity)/(arc OX)

is a constant quantity independent of OX.

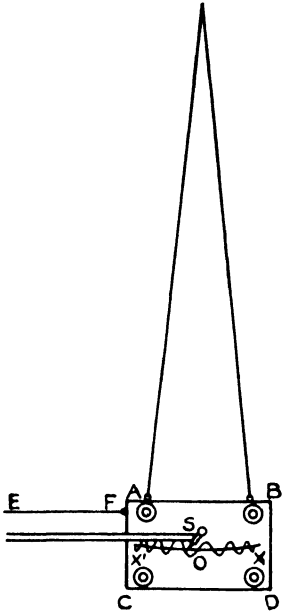

Fig. 31*a*.

Procedure.

(1) Set up the apparatus as in figs. 31*a* and 31*b*, the wave paper being attached to the board by drawing pins.

Care should be taken that the spring is set up so as to vibrate accurately in a vertical plane, and also that the brush records a line of uniform thickness as the pendulum swings back and forth.

(2) It was found convenient to put the spring in the U-shaped holder used in the trolley experiments (e.g. Experiment XXI.), and to fasten the latter to a vice attached to a small table (fig. 31*b*).

The position of this vice must be adjusted so that the brush is in contact with a point about mid-way along the board (the point O) when the pendulum is at rest.

(3) Attach the thread EF to the board by a ring-screw, and pass it over a pulley, which may be fixed in the same vice.

Attach to the thread a 1/10-lb. hanger and sufficient slotted weights to hold the board aside the desired amount.

(4) By lifting the weight up quickly with the hand, cause the brush to draw the circular arc XOX′ on the paper. The pendulum must be stopped at the point X′, and brought back to rest at its central position (the point O): care being taken not to disturb the setting of the brush.

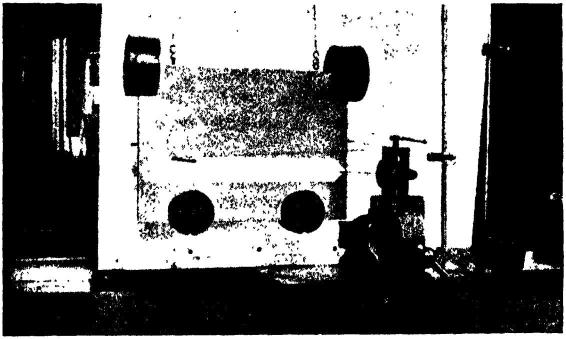

Fig. 31*b*.

(5) Mark the position of the point O on the circular arc.

(6) Draw back the pendulum until the brush is again at the point X. X is an arbitrary point which should be varied in successive observations by altering the weight on the hanger.

(7) With one hand draw aside the spring about 2 cm., and with the other apply a flame to the thread EF.

Let go the spring at the instant when the thread is severed.*

---

* An alternative is to dispense with the weight, and to attach the thread—after passing it over the pulley—to the side of the board. This is the arrangement actually shown in Fig. 31*b*.

The pendulum must be stopped at the opposite end of its swing to prevent a second wave being drawn on the paper.

(8) Remove the paper and measure carefully with a compass and ruler the complete wave-length on either side of O. Take the average of these two as the mean-position-velocity V.

In the first observation detailed below

Average of the two wave-lengths = 2·78 cm.,
Hence V = 2·78 cm./vibn.
Also amplitude OX = a = 20·2 cm.

(9) Repeat the experiment with a different value for OX, i.e. by altering the weight on the hanger.

The second observation detailed below gave
V = 2·55 (cm./vibn.) and a = 18·46 cm.

At least five different observations should be made.

(10) By altering the number of kilogramme weights screwed to the board of the pendulum, and obtaining a new set of observations, show that the ratio

$$V/a$$

is independent of the mass of the pendulum.

Table XXXI. below gives the results of a set of observations for the same mass of pendulum :-

TABLE XXXI.

Mean-position-velocity is proportional to amplitude of arc.

Obsn. No.	1.	2.	3.	4.	5.	6.
a............	20·20	18·46	17·30	16·09	14·45	11·50
V..........	2·78	2·55	2·40	2·24	2·00	1·60
V/a...	0·138	0·138	0·139	0·139	0·138	0·139
°/₀ diff. from mean	− 0·361	− 0·361	+ 0·361	+ 0·361	− 0·361	+ 0·361
Mean..**0·1385**						

EXPERIMENT XXXII.

Object of Experiment:

To determine the value of *g*, the acceleration of a body falling freely to earth under gravity. (An alternative method to that of Experiment XXX.)

Apparatus:

Light wooden pendulum (fig. 32) pivoted at one end at right angles to its length, 2 Capstan stands, 2 Capstan blocks, brass or lead plummet, thread, 2 thin brass rods to form guides for the thread, stop-watch (or chronograph), duster, ink mixed with glycerine (such as is used for a barograph).

(This experiment, while not capable of yielding results of great accuracy, has the merit of simplicity.)

Outline of Method:

The instrument consists of a light wooden pendulum swinging on a pivot at one end at right angles to its length, the pivot being attached to a block on a Capstan stand (fig. 32).

A thread is attached to the pendulum about half-way down; and, after passing round the horizontal rod shown projecting from the right hand Capstan stand in the figure, passes over another guide fixed above the top of the pendulum, and carries at its other end a brass (or lead) plummet.

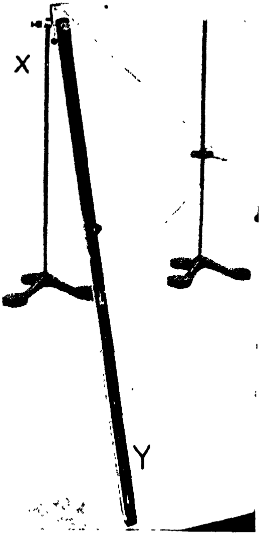

FIG. 32.

Initially, the pendulum is allowed to hang freely at rest, and the thread is adjusted—by moving both guides—so that the edge of the plummet just touches the pendulum near the top.

The pendulum is now pulled aside through a small angle (2 to 3 degrees).

The rim of the plummet is then inked, and a mark X is made on the pendulum by swinging the plummet to contact with it.

Next the thread is burnt, so that the plummet is released, and the pendulum set in motion simultaneously.

When the pendulum reaches its lowest position, the plummet collides with it, making an ink mark Y. XY is thus the distance fallen by the plummet from rest in a quarter-vibration of the pendulum.

The period of vibration of the pendulum is determined separately with a stop-watch, or other time-piece.

Hints for Performing the Experiment.

(1) The most convenient shape for the plummet is that of two equal cones with their bases together. The rim of the bases then gives a sharply defined contact with the pendulum.

(2) The stand from which the pendulum is hung should be placed at the edge of the table, with two of its feet flush with the edge.

(3) The pendulum must be arranged to swing accurately in a vertical plane, otherwise the falling plummet will fail to make contact with it at the lower end. The best way to test for this is first to use the plummet and thread as a plumb-line, suspending it from the guide above the top of the pendulum. If the latter is found to be slightly out of the vertical, the stand must be suitably tilted—e.g. by placing pieces of thin cardboard under a foot.

(4) The guide above the top of the pendulum must be adjusted so that the plummet hangs with its rim just in contact with the pendulum, when the latter hangs freely; **both at the top where the fall starts, and at the bottom in the neighbourhood of the place where the collision is expected to occur.**

Delicate contact in the lower position is just as important as at the top, for here (at the bottom) the plummet is travelling quickly; so that if the collision actually occurs slightly before or after the calculated quarter-period, the value of g as obtained from the observations will be appreciably affected. (See Exercises at the end of the Experiment.)

(5) After the adjustment of the guide above the pendulum, the other guide should be adjusted so that the plummet string holds the pendulum aside through an angle of between 2 and 3 degrees. For this purpose, the free end of the thread is fastened to the pendulum about half-way down. The plummet must be sufficiently heavy to keep the pendulum held aside in the position in which it is set; hence the necessity for the pendulum to be as light as possible.

(6) In order to record the starting-point X of the plummet's fall, and Y, the point at which it collides with the bottom of the pendulum, the rim of the plummet should be inked. It is well to mix some glycerine with the ink, to prevent it drying too quickly; or else to use special non-drying ink such as is employed in a barograph.

(7) A duster, or a small coconut mat, should be placed under the pendulum to catch the plummet when it falls.

(8) The thread should be burnt with a taper between the guides, and as near as possible to the top of the pendulum.

(9) Disengage the loose end of the thread, and take the time of 20 or more complete oscillations of the pendulum, with a stop-watch (or chronograph).

Hence find the average time t of a quarter-period;
Then, if XY = s feet, g is given by

$$s = \tfrac{1}{2}gt^2.$$

(10) Table XXXII. below, which gives the details of an actual experiment, shows the method on which the results should be tabulated:—

TABLE XXXII.

Experimental Results.

Expt. No.	1	2, etc.
s (ft.)	3·24	
t (sec.)	0·45	
t^2	0·2025	
$2s/t^2$ $(= g)$	32·00	
°/ₒ difference from true value	– 0·60	

Exercises.

1. In an experiment of the above kind, the pendulum was pulled aside through a distance of 1·5 inches at the bottom. The time of a quarter-period was observed to be 0·45 sec., and the distance XY was recorded as 3·7 feet. It was accordingly assumed that the plummet had fallen a distance of 3·7 ft. in 0·45 sec. What value of g does this give?

(Ans. $g = 36·55$.)

2. It was afterwards noticed, that, on lowering the plummet to the bottom of the pendulum, there was a gap of 1/10 in. between the two when the latter was hanging freely. The true time of flight of the plummet thus exceeded a quarter-period.

Assuming that the velocity of the pendulum near its lowest position is approximately uniform, calculate the true time of flight of the plummet, and hence deduce the proper value of g which these observations should have yielded.

(Ans. 0·48 sec.; 32·11 ft./sec.^{2})

* EXPERIMENT XXXIII.

VECTOR BALANCE.

Object of Experiment:

To test experimentally the principle of the Conservation of Momentum for the impact of two spheres, i.e., to show that

the total momentum of the system before impact = the total momentum after impact.

Apparatus:

Goodwill Vector Balance† *consisting of two steel spheres fitted with pointers, and suspended by adjustable threads of length about* 10 *ft. from a special bracket; small stool, drawing-board, small electromagnet, Capstan stand and arrangement of clamps to hold the electromagnet, accumulator, switch, fuse, small adjustable resistance (about* 4 *ohms), Morse key,* 6 *wire leads.*

Theory of the Vector Balance:

This instrument makes use of the principle established in EXPERIMENT XXXI., viz., that the mean-position-velocity of a simple pendulum is directly proportional to the amplitude of the arc of swing; and further, that this factor of proportionality is independent of the mass of the pendulum.

Each sphere and its supporting thread is, in fact, a simple pendulum; and, by the above principle, the maximum displacements (amplitudes) of a sphere before and after impact form a measure of the velocities of that sphere immediately before and after the impact.

The strings supporting the spheres are adjusted so that when they are at rest, the latter are just in contact with their line of centres horizontal.

Let one sphere now be pulled aside a distance x, and then be allowed to impinge on the other, and let y be its maximum displacement after the impact; then x and y may be taken as a measure of the sphere's velocities just before and after impact.

† For further information as to the uses of this instrument, see Goodwill's *Elementary Mechanics* (Clarendon Press).

The velocities of the second sphere are similarly obtained.

Fig. 33*a*, originally designed for the more extensive experiments described in Goodwill's *Mechanics*, shows a general view of the 3-sphere Vector Balance; while fig. 33*b* shows the author's arrangement of apparatus for the experiment to be described below.

Outline of Method:

The experiment here described is the simplest one which can be performed with two spheres; viz., the impact of a moving sphere with a stationary one.

After adjusting the threads until the spheres hang just in contact with their line of centres horizontal, one sphere (mass m_1) is pulled aside a measured distance x_1 from its mean position, and held there by the electromagnet; meanwhile the other sphere (mass m_2) hangs at rest in its mean position.

The distance x_1 is recorded on a sheet of paper placed below and just clear of the pointers of the spheres.

m_1 is then released by switching off the current, so that it impinges directly on m_2.

The maximum displacements y_1, y_2 of the two spheres after impact are also recorded on the same paper.

Thus x_1 and y_1 measure the velocities of m_1 immediately before and after impact, while y_2 measures that of m_2 immediately after impact.

Then, if the displacements are all in the same direction, it is the object of the experiment to test whether

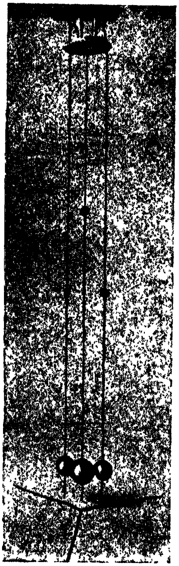

FIG. 33*a*.

$$m_1x_1 = m_1y_1 + m_2y_2,$$

or, to put it in a form more convenient for calculation, whether

$$m_1(x_1 - y_1) = m_2y_2.$$

Procedure.

(1) The bracket (fig. 33*a*) from which the spheres are suspended must be sufficiently high up to allow the suspending threads to be a considerable length. In the experiment

described here the bracket was fastened to a rafter near the roof, and the suspending thread were about 10 ft. long (actually 308 cm. to the bottom of the pointers). This condition is necessary in order that fairly large displacements may be obtained, while at the same time the angular displacements are sufficiently small for the motions of the spheres to be regarded as Simple Harmonic. (See Appendix II., Notes 2 and 3.) That is to say, the angular displacements should all be less than 4°.

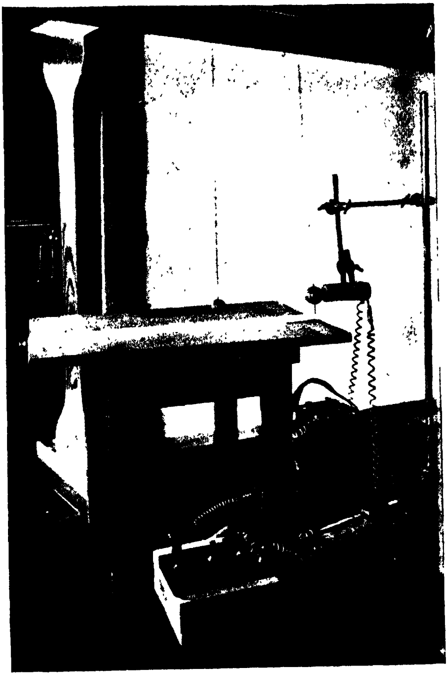

FIG. 33*b*.

(2) The Vector Balance bracket is fitted with slots round a circle which allow of the threads being set at different distances apart. The threads should be set at such a distance

apart that the two spheres, when at rest, hang just in contact with one another.

(3) Take a piece of paper which is the length of the drawing-board (e.g. a piece of the special wave-tracing paper used for EXPERIMENT XXXVII.). Pin it to the board, and rule a pencil line along it.

Place a small stool under the spheres, and on it place the drawing-board and paper.

(4) Adjust the lengths of the threads so that, while the line of centres of the two spheres is horizontal, the pointers are just clear of the paper.

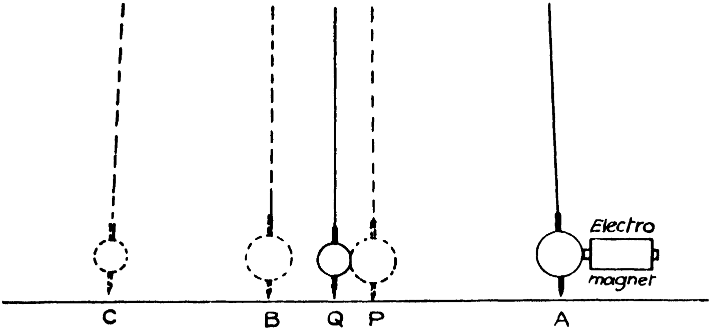

FIG. 33c.

(5) Now move the drawing-board so that the pointers are both just vertically above the line.

With a pencil mark points vertically under the pointers as they now hang at rest.

Let P be the point under m_1, and Q that under m_2 (fig. 33c).

(6) Set up an electromagnet, as shown in fig. 33c, with which to hold aside the sphere m_1.

Only a small magnet is needed. The one shown in the figure was made from a wooden reel about 3 in. long, round which was wrapped a few layers of insulated wire —about No. 14 gauge. Down the hole in the reel was placed a piece of iron rod slightly longer than the reel.

In order that the slight residual magnetism still present in the rod when the current was switched off might not prevent the release of the sphere, a little piece of paper was stuck on the end of the rod.

(7) In the experiment described, the magnet was connected to a Morse key, 4-volt accumulator, ordinary switch, fuse, and adjustable resistance of about 4 ohms. The method of producing the current may of course be altered to suit requirements, but in any case it is convenient to have a Morse key to release the sphere, and also an ordinary switch with which to turn off the current when not required, without disconnecting the Morse key.

(8) With a pencil mark a point A on the ruled line a given distance, say 15 cm. from P.

Now pull back the sphere m_1 until its pointer is vertically above A, and adjust the position of the electromagnet so that the sphere is held there.

Thus AP (= 15 cm. = x_1) is the measure of m_1's velocity immediately before impact with m_2.

(9) With the left hand press the switch of the Morse key; this key should be placed conveniently near the drawing-board (see fig. 33*b*).

The sphere m_1 will now swing down and impinge on m_2.

(10) With a pencil mark the points B and C, which indicate the maximum displacements of m_1 and m_2 respectively, after the impact.

Thus BP = y_1, and CQ = y_2 are the measures of the velocities of m_1 and m_2 respectively, after the impact.

(11) It will probably be found difficult to record *both* of the points B and C in one experiment, owing to the shortness of time available for moving the hand from the one to the other.

To get over this difficulty, the experiment should be repeated a number of times, concentrating attention upon B and C alternately. Then take the mean of all the observations.

The following procedure was adopted in the experiment described here:—

(*a*) After the first impact the position of C was marked. As

m_1 was about twice the mass of m_2 (see **Experimental Results** below), CQ was much bigger than BP.

(*b*) A narrow strip of paper about an inch long was folded into the shape of an "L", and placed over the point C.

The experiment was then repeated, and this time the position of B was marked; immediately afterwards it was noticed whether m_2 came exactly up to, fell short of, or pushed slightly forward, the little L-shaped piece of paper.

In this way it was ascertained whether there was any change in the position of C for this second experiment.

(*c*) This pair of experiments was repeated four times altogether, thus providing four values of y_1, and four of y_2.

Experimental Results.

$m_1 = 129{\cdot}82$ gm., AP $= x_1 = 15$ cm.,

$m_2 = 66{\cdot}17$ gm., BP $= y_1 = 5{\cdot}95$ cm. (average of 4 observations),

CQ $= y_2 = 17{\cdot}66$ cm. ,, ,, ,,

We have to test whether $m_1(x_1 - y_1) = m_2y_2$.

In this case $x_1 - y_1 = 9{\cdot}05$,

and $m_1(x_1 - y_1) = \mathbf{1175}$,

while $m_2y_2 = \mathbf{1169}$.

Mean of these $= 1172$

Deviation of either from the Mean $=$ **0·26 per cent.**

(The distance from the suspending bracket to the paper was 308 cm., so that the largest angular displacement obtained was

$$17{\cdot}66/308 \text{ of a radian} = 0{\cdot}0573 \text{ radian}$$
$$= 3^\circ\ 17',$$

which is well within the range for which the oscillations may be regarded as Simple Harmonic.)

Experimental Exercise.

Determine the *coefficient of restitution* (see p. 98) for the two spheres used in this experiment.

EXPERIMENT XXXIV.

MEASUREMENT OF BRAKE-HORSE-POWER BY FRICTION DYNAMOMETER.

Apparatus:

Electric motor fitted with fly-wheel having a broad rim, rope brake, kg. hanger and slotted kg. weights, lb. hanger and slotted lb. weights, revolution counter, ammeter, voltmeter, spring-balance, stop-watch.*

Method :

A rope brake (described in more detail in the Experimental Section below) is fitted on the fly-wheel (fig. 34*a*.)

One end of the brake is anchored, through a spring-balance, to a fixed hook, while the other end carries a suitable weight (W lbs. wt.)

When the wheel is stationary, the tension in the rope is equal on either side of the wheel, and the spring-balance reading should be the same as the weight W.

If, however, the wheel be rotated in the direction of the arrow, the friction between the rope and the rim of the wheel will carry the former a short way back with it: and if the wheel revolve at a constant speed, a steady state will quickly be reached at which the difference between W and the spring-balance reading S, viz.

$$W - S$$

is the measure of the frictional force against which the wheel is revolving.

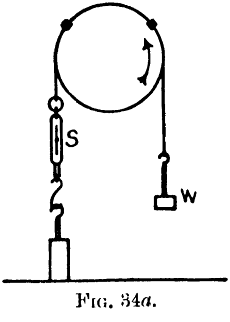

Fig. 34*a*.

Let the wheel revolve at N revolutions per second, and let R ft. be its radius.

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* As far as the measurement of Brake-Horse-Power goes, the experiment may be performed with a wheel rotated by any means.

Then the work done per second against the resistance of the brake is

$$2\pi RN(W - S) \text{ foot-pounds.}$$

Thus the Brake-Horse-Power at this speed is

$$2\pi RN(W - S)/550. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

Let V and A be the readings of the voltmeter and ammeter respectively, then

$$\text{I.H.P.}^* = VA/746 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$

and the ratio of (a) to (b) gives the *efficiency* of the motor at that speed.

Experimental.

The experiment whose details are given below was performed with a Direct-current Shunt-wound motor running from 220 volt Mains.

For this motor a convenient form of brake was found to consist of two pieces of cord (e.g. blind cord) threaded through holes in two pieces of wood, each shaped as in fig. 34b, the projections at either end being designed to prevent the wood, and therefore the rope, from slipping off the wheel.

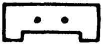

FIG. 34b.

The loose ends of the cord are knotted together, and one end is attached to a spring-balance, which in turn is attached to a rigid hook: the other end of the cord is attached to a hanger carrying the weight W.

To avoid the necessity of fixing a hook in the floor, it was found convenient to use instead the hook of a kilo. hanger, which was loaded with sufficient weights to prevent it being moved by the weight W.

Procedure.

(1) Place the ammeter in the electric circuit to the Mains, and the voltmeter across the brushes of the motor.

(2) Run the motor for a few moments to ascertain the direction of rotation of the wheel.

(3) Attach one end of the rope brake to the spring-balance, and the spring-balance to the fixed hook (fig 34a).

To the other end of the brake attach the 1-lb. hanger, and fit the whole on to the wheel, taking care that the weight W acts so as to oppose the rotation of the wheel.

---

* That is, the *Indicated Horse-Power*—the power put into the machine.

(4) With one hand lift up the 1-lb. hanger so as to relieve the pressure of the brake on the rim of the wheel:
With the other hand switch on the current.
Next, let the hanger gradually down.

(5) Having noted the spring-balance reading, and the reading of the voltmeter and ammeter, press the revolution counter against the axle of the wheel,* and with a stop-watch determine the number of revolutions per minute N. (If the revolution-counter is fitted with a spring contact, an ordinary watch with a seconds hand is sufficient.)

Details of an actual experiment:—

$2R = 5\cdot63$ in.,
$W = 3$ lbs. wt.,
$S = 1\cdot2$,, ,, $W - S = 1\cdot8$ lbs. wt.
$V = 200$ volts,
$A = 0\cdot89$ amperes,

Initial reading of rev. counter = 2370
reading after 20 seconds = 1705
Difference = 665

Thus $N = 33\cdot25$ revs. per sec.

Hence, from formulæ (*a*) and (*b*) above,

$$\text{B.H.P.} = 5\cdot63 \times \pi \times 33\cdot25 \times 1\cdot8/(550 \times 12).$$

i.e. **B.H.P = 0·1604** (roughly $\frac{1}{6}$ H.P.)

and $\text{I.H.P.} = 200 \times 0\cdot89/746.$

i.e. **I.H.P. = 0·2382**

Taking the ratio of these two, we have the efficiency

e = **67·3 per cent.**

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* At this point it is convenient to summon a second observer to read again the spring-balance, the voltmeter and ammeter, as the pressure of the revolution-counter against the axle may slightly alter these readings.

Theoretical Exercises suggested by Experiment XXXIV.

(1) Prove that the work done per revolution by a couple of moment M is $2\pi M$.

(2) Obtain the expression for the Brake-Horse-Power of the motor in the above Experiment by considering the work done per second by the couple which drives the motor.

* EXPERIMENT XXXV.

BRAKE-HORSE-POWER OF AN ELECTRIC MOTOR MEASURED BY A REACTION METHOD.

Apparatus:

Electric fan hung from a large beam supported as in fig. 35b, 250-volt voltmeter, 0-1 ampere ammeter, 3 Capstan stands and blocks, 2 pulleys and 1 small hook, 6 kilo. weights to be placed on the feet of the stands, 2 4-lb. spring-balances, revolution-counter.

(This list of apparatus applies to the particular experiment shown in fig. 35b. A modification for different conditions will suggest itself to the student.)

Method:

The experiment can be done most conveniently with an electric fan enclosed in a symmetrical case, and capable of being suspended from a point on the axis of rotation, so that the latter is vertical.

If the fan be suspended freely in this way, and the motor started, the couple which drives it gives rise to an equal and opposite reaction on the framework, causing rotation of the latter in the opposite direction.

The B.H.P. of the motor is determined by measuring the couple needed to prevent this backward rotation of the frame.

To effect this measurement, a rod AB (fig. 35*a*) is fixed to the frame at right angles to the axis, and the motor is suspended from a beam by a cord or chain so that the axle hangs down vertically in the line of the chain.

At points A and B of the rod at equal distances R from the axis are attached the horizontal strings AC, BD, passing over the pulleys Y, Z to the spring-balances S_1, S_2; the direction of the strings being such as to oppose rotation of the frame.

Let the motor be now set in motion at a speed of N revolutions per minute; and let F_1, F_2 be the tensions in the strings, as measured by the spring-balances.

Then the work done by the couple which holds the frame stationary is, in N revolutions, or 1 minute

$$2\pi RN(F_1 + F_2) \text{ foot-pounds.}$$

Thus $$\text{B.H.P.} = 2\pi RN(F_1 + F_2)/33000 \ldots\ldots\ldots\ldots \quad (a)$$

In order to measure the I.H.P., an ammeter must be inserted in the circuit to the battery or Mains driving the motor; and a voltmeter placed across the brushes of the motor.

If A and V be the readings of the ammeter and voltmeter respectively, then

$$\text{I.H.P.} = VA/746 \ldots\ldots\ldots\ldots\ldots\ldots \quad (b)$$

while the efficiency e is the ratio of (a) to (b).

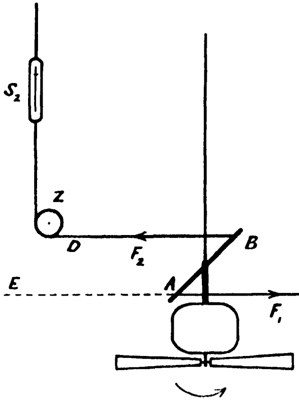

FIG. 35*a*.

Hints for Performing the Experiment.

(1) First ascertain the direction of rotation of the motor.

(2) The strings AC, BD must be horizontal. The most convenient value of R, their distance from the axis, is best found by trial; if it is too large the spring-balance readings will be inconveniently small.

(3) The height of the spring-balances must be so adjusted that the rod is at right angles to the strings at the particular speed of rotation which is being employed.

(4) On stopping the motor, it will be found that the frame kicks

back in the direction in which the motor was revolving. As this is inconvenient, it should be prevented by the opposing string AE (fig. 35*a*) adjusted so as to be taut for the zero reading of the spring-balances. When the motor is rotating, this string is slack, and does not affect the observations.

Details of an experiment with the electric fan shown in fig. 35*b* :—

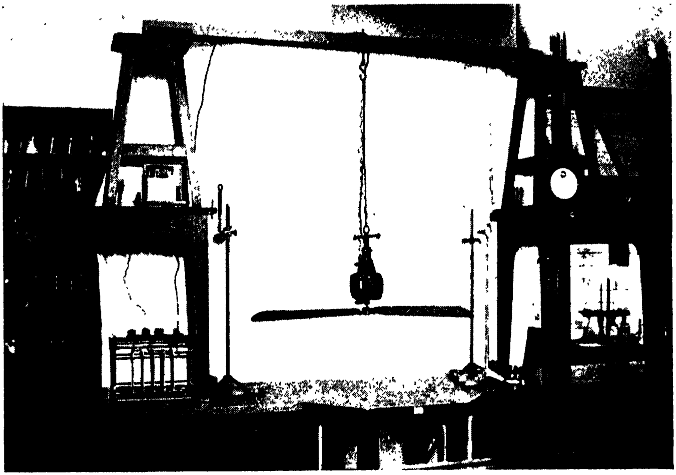

FIG. 35*b*.

$R = 1{\cdot}5$ in. $= \frac{1}{8}$ ft.,
$F_1 = F_2 = 0{\cdot}7$ lbs. wt.,
$N = 228$ r.p.m.,
$V = 30$ volts,
$A = 0{\cdot}26$ amperes.

Length of fan blades from axle to tip = 18 in.
Average width = 5 in.

The blades were set at an angle of about 70° to the axis of rotation.

Hence

$$\text{B.H.P.} = 2\pi \times 228 \times 1{\cdot}4/33000 \times 8.$$
$$= 0{\cdot}0076.$$
$$\text{I.H.P.} = 30 \times 0{\cdot}26/746.$$
$$= 0{\cdot}0105$$
$$e = 72{\cdot}5 \text{ per cent.}$$

* EXPERIMENT XXXVI.

CENTRIFUGAL FORCE.

Apparatus:

Cussons' Centrifugal Force Machine (fig. 36b), stop-watch or clock. (An electric motor with which to rotate the machine is convenient, though the operation can be done quite satisfactorily with the hand-wheel.)

Description of Instrument:

This instrument is of the weight-lifting type, that is to say, the centrifugal force arising from the rotation of one or more masses is employed to lift a known weight, whose magnitude, under certain conditions, affords a measure of this centrifugal force.

Several types of centrifugal force machine exist; there is, for instance, the Robinson machine †, in which a rotating mass is caused to lift a weight through the intermediary of a swivelled string.

The Cussons' machine employs a different method for raising the weight, which will be grasped from fig. 36*a*.

M, M are the rotating masses, which are connected to the platform TTTT through the pair of bell-crank levers ACDEG, whose pivots are at C, C.

Each graduated rod AB, at any point of which M may be clamped, is pivoted to its lever at A, while it is free to slide horizontally in the upright V. The pivot at A is easily removed, allowing the mass M to be taken off, and a different one substituted.

Confining attention first to the right-hand side of the rotating system as shown in fig. 36*a*: the arms AC, CD of the lever being equal, and at right angles, a horizontal force acting at A will, in all positions of the lever, give rise to an equal vertical force at D. Similar arguments apply to the left-hand side.

---

† See below, p. 162.

Resting on the elbow D of the lever is a sleeve S of known weight (2 lbs.)*, which closely fits the axle YY, but is free to slide up and down it. In order to increase the effective weight of S, a number of iron rings of known weight (1 or 2 lbs.) may be placed upon it.

If, now, the platform TTTT and its attachments be made to revolve about the axle YY by means of the pulley P, the centrifugal force F', due to the rotation of each mass M, gives rise to a radially outward force F in each rod AB. We thus have a vertically upward force $2F$, due to the centrifugal action of both † masses, tending to make the sleeve S and its load rise up the axle.

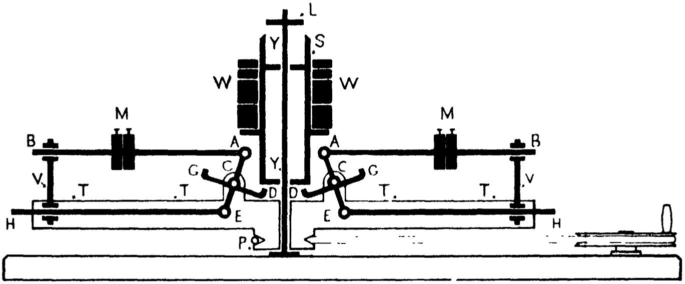

FIG. 36*a*.—Diagram of Cussons' Centrifugal Force Machine.

In any position, then, in which the sleeve and its load is held "floating" on the axle, the total weight W is a measure of the centrifugal force arising from the motion of the two masses M, i.e.

$$W = 2F.$$

---

* Actually, there is between the sleeve and the lever a pair of washers, separated by ball-bearings, in order to reduce friction.

† If it be desired to utilise the weight-lifting effect of *one* mass M only, this result may be secured by clamping one of the rods AB by means of a screw fitted for the purpose in the upright V. One side of the rotating system is thus thrown out of action, while at the same time the centre of mass of the rotating system is still (approximately) on the axis YY.

(In order to eliminate the contribution to F which would arise from the centrifugal action of the rod AB itself, there is fitted below the platform the equal rod EH, pivoted at E and free to slide in a similar manner to AB. This ensures that the force F is due to the centrifugal action of M only.)

Now the adjustment will be most sensitive in the position in which the arm AC of the lever is vertical (why?).

In order that the attainment of this position may be easily recognised when the instrument is in motion, the axle YY is fitted with a small collar L, so placed that when the top of the sleeve S is level with the top of the collar, the arm AC is exactly vertical.

When the correct speed of rotation has been reached which causes the loaded sleeve to float in this position, the revolution-counter is read at the beginning and end of an interval of time measured with a stop-watch or clock.

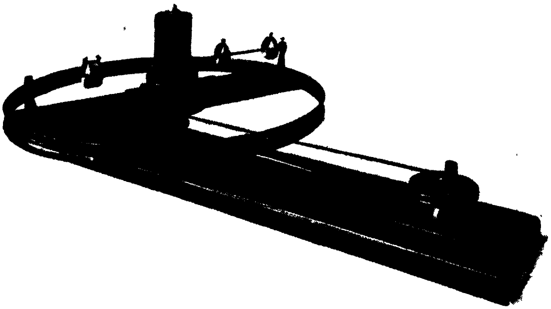

FIG. 36*b*.

Theoretical Discussion.

Let N = no. of revolutions per second of rotating system,
R = distance of either mass M from axis YY,
v = peripheral velocity of these masses.

It is, then, the object of the experiment to test the validity of the relation

$$Wg = 2Mv^2/R. \quad \left\{\begin{array}{l}\text{poundals}\\ \text{dynes}\end{array}\right\} \cdots\cdots\cdots(1)$$

which is the mathematical expression of the fact that the "floating" weight W balances the centrifugal force arising from the rotation of the two masses M.

It is first necessary to reduce the relation (1) to a form involving quantities which may be directly measured.

In N revolutions, or one second, the centre of mass of M moves through a distance

$$2\pi RN \begin{matrix}\text{ft.}\\ \text{cm.},\end{matrix}$$

$$\text{thus } v = 2\pi RN,$$

so that the relation to be tested experimentally is

$$Wg = 8\pi^2MR^2N^2/R = 8\pi^2MRN^2,\text{*} \ldots\ldots\ldots\ldots \quad (2)$$

which may also be written

$$W = kMRN^2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$\text{where } k = 8\pi^2/g.$$

Experimental Procedure.

The relation (3) involves four experimentally determinate variable quantities, W, M, R, N. Hence the following four experiments are suggested:

Experiment XXXVIa.

Test whether W is directly proportional to M, when R and N are constant.

Experiment XXXVIb.

Test whether W is directly proportional to R, when M and N are constant.

Experiment XXXVIc.

Test whether W is directly proportional to N^2, when M and R are constant.

Experiment XXXVId.

Test whether, when all four quantities vary, the expression

$$MRN^2/W$$

is constant and equal to

$$g/8\pi^2.$$

---

* The relation (2) may also be expressed in terms of angular velocity (radians per second), for $\omega = 2\pi N$, so that (2) becomes

$$Wg = 2M\omega^2R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2a)$$

Hints for the Performance of these Experiments.

Experiment XXXVIa.

Set two equal masses M of $\frac{1}{4}$ lb. at equal distances from the axis, and, after selecting a suitable value for the balance-weight W, find by trial the speed of rotation which causes W to "float" at the correct level.

Keeping the speed at this value, take the time of 100 revolutions with a stop-watch.

Next, increase M to $\frac{1}{2}$ lb. (without altering its distance from the axis), and adjust W until the correct level of flotation is attained for the same speed of revolution as before.

Repeat the observation, increasing M by steps of $\frac{1}{4}$ lb., until the highest feasible value of W has been reached.

The following results obtained for R = 8 in. will serve as a guide for the student:—

TABLE XXXVIa.

Relation Between W and M (R and N Constant).

Obsn. No.	1	2	3	4
M (lbs.)........................	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
W (lbs.)........................	2	4	6	8
Time (secs.) of 100 revs.	45·0	45·1	45·0	45·2

It is here unnecessary to evaluate N, as it is seen by inspection to be constant within $\frac{1}{2}$ per cent.

The observations thus show that, within the limits of experimental error, W is directly proportional to M.

A large number of variations of this experiment may obviously be performed by choosing a different value for R, and a different initial value for W.

Experiment XXXVIb.

Choosing for M any one of the individual values used in Experiment XXXVIa, set it at radius R = 4 in.; and then, choosing a convenient initial value for *W*, find by trial the speed of revolution which causes *W* to float at the correct level. Keeping the speed at this value, take the time of 100 revolutions with a stop-watch.

Repeat the observation, with R increasing by steps of 1 in , up to the limit allowed by the rod AB.

The following are a few observations for the case when $M = 1$ lb.:

TABLE XXXVIb.

Relation Between *W* and *R* (*M* and *N* Constant).

Obsn. No.	1	2	3	4
R (in.)........................	4	6	8	10
W (lbs.)......................	4	6	8	10
Time (secs.) of 100 revs.	45·0	45·2	45·0	45·0

Here *N* is evidently constant within $\frac{1}{2}$ per cent. Inspection shows that *W* is directly proportional to *R*.

As before, a variety of different observations will be obtained by different choices for *M*, and the initial value of *W*.

Experiment XXXVIc.

Choose particular values of *M* and *R* (which are to remain unaltered throughout the experiment); and take $W = 2$ lbs., initially.

Find by trial the speed of revolution at which *W* " floats " at the correct level, and at this speed take the time of 100 revolutions with a stop-watch.

Repeat the observation with *W* increasing in steps of 1 lb. up to, say, 10 lbs.

The following are a few such observations for the case when $M = \frac{1}{2}$ lb., $R = 8$ in.:

TABLE XXXVIc.

Relation Between *W* and *N* (*M* and *R* Constant).

Obsn. No.	1	2	3	4	5
W (lbs.)........................	2	4	6	6	10
Time (= T secs.) of 100 revs.	64·0	45·1	36·9	31·9	28·6
N (= $100/T$)........................	1·563	2·217	2·710	3·135	3·497
N^2........................	2·443	4·915	7·344	9·828	12·23
N^2/W........................	1·222	1·229	1·221	1·228	1·223

The mean value of N^2/W is 1·225, from which the extremes differ by less than ½ per cent.; hence, within this margin, W is directly proportional to the square of N.

Experiment XXXVId.

Calculate the correct value of $g/8\pi^2$ for the laboratory in which the experiments are being performed. Retabulate the observations provided by the first three experiments so as to show the value of

$$MRN^2/W,$$

as these four quantities change progressively from the smallest to the largest values which it is feasible to allow them to assume.

Where gaps occur in the table, additional observations must be made.

Finally, calculate the average value of MRN^2/W, and its percentage difference from $g/8\pi^2$.

The following are a few observations tabulated in this manner:

TABLE XXXVId.

Comparison Between MRN^2/W and $g/8\pi^2$.

Obsn. No.	M (Lbs.)	R (Ft.)	W (Lbs.)	Revs.	Time (Secs.)	$N = \frac{\text{Revs.}}{\text{Time}}$	N^2.	MRN^2/W.
1	$\frac{1}{4}$	$\frac{2}{3}$	2	400	180·0	2·222	4·937	0·411
2	$\frac{1}{4}$	$\frac{2}{3}$	6	300	78·0	3·816	14·80	0·411
3	$\frac{1}{4}$	$\frac{2}{3}$	10	400	80·4	4·976	24·75	0·413
4	$\frac{1}{2}$	$\frac{1}{3}$	4	300	96·0	3·125	9·766	0·407
5	$\frac{1}{2}$	$\frac{2}{3}$	4	200	90·2	2·217	4·915	0·410
6	$\frac{1}{2}$	$\frac{2}{3}$	8	150	48·0	3·125	9·766	0·407
7	$\frac{3}{4}$	$\frac{2}{3}$	6	100	45·0	2·222	4·937	0·411
8	1	$\frac{1}{3}$	4	100	45·0	2·222	4·937	0·411

To 3 signif. figs.—Av. value of $MRN^2/W = 0·410$
,, ,, ,, .— ,, ,, $g/4\pi^2 = 0·410$

The average experimental value of MRN^2/W afforded by these experiments is thus equal to the value of $g/8\pi^2$ correct to 3 significant figures.

The last three experiments (*b*, *c*, and *d*) may also be performed with a Robinson Centrifugal Force machine,* the features of which will be grasped from figs. 36*c* and 36*d*.

In this instrument, the balance-weight W is lifted by the centrifugal action of a single mass M. The two are connected by a string, which is swivelled at L to prevent rotation of the upper portion.

The function of M′ is to keep the centre of mass of the rotating system on the axis OY.

---

* Made by Messrs. W. G. Pye & Co. of Cambridge.

In order to observe the distance of M from the axis while the machine is in motion, the pointer X, attached to the upper part of the string, indicates on the adjustable scale S the distance of M from the axis of rotation.

As in many of these instruments only the rod CD is graduated, the most convenient way to set the scale to read correctly is as follows: clamping M′ so that its centre is a given distance, say, 4 in. from the axis, fit a pair of parallel-jaw calipers in between M′ and the side CC′ of the box AA′ CC′; then transfer the calipers to the other side, and slide the mass M up to contact. The tension in the string due to the weight W (the hanger alone is sufficient) will keep M in position while the scale is adjusted so that the pointer reads 4 in.

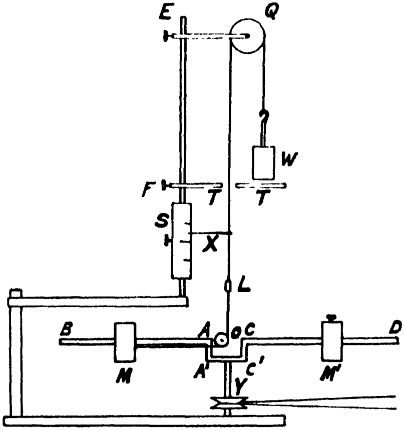

FIG. 36c.

The scale will now read correctly for all distances of M from the axis.

Care should be taken that W does not overwind on the pulley Q, causing breakage of the string. To guard against this, it is well to rest the left hand lightly on the pulley, while revolving the machine with the right hand, until the correct speed of revolution has been attained.

In the hands of a careful experimenter, this apparatus is capable of giving results correct to within 1 per cent.

Theoretical Exercises suggested by Experiment XXXVI.

1. Calculate the values of the angular velocity ω for the observations in Table XXXVId. (Either your own table, or that given above.) Hence recast these results so as to show a comparison between W and

$$2M\omega^2R/g.$$

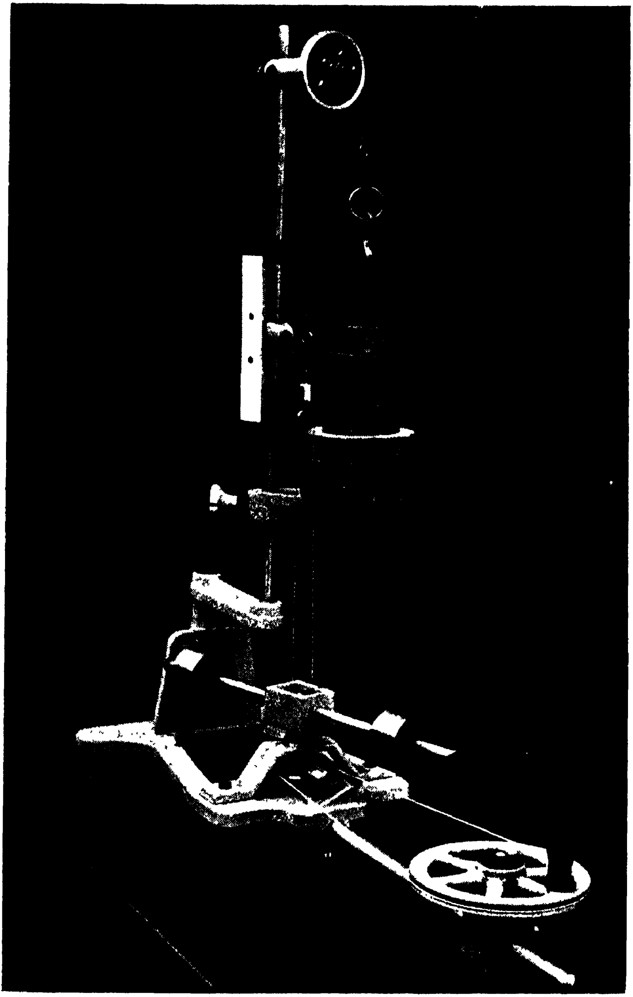

FIG. 36*d*.

2. If, in an experiment with the Cussons' machine, the balance-weight W be kept floating so that the top of the sleeve S is $\frac{1}{16}$ in. below its correct level (i.e. level with the *bottom* of the collar L), calculate the percentage errors introduced into the estimate of the centrifugal force for the cases when R = 4, 6, 8, 10 inches, respectively.

3. A horizontal rod of length 1 metre is fixed at its middle point to a vertical axis about which it can rotate. The rod carries on either side identical blocks, each of mass $\frac{1}{2}$ kg., which can slide along the rod with negligible friction. The blocks are fastened together with a piece of string, so that their centres of mass are 20 cm. apart, and equidistant from the axis.

   If the rod be now rotated with uniform angular velocity of 2 radians per second

   (i) Calculate the tension in the string, stating the units in which it is measured;

   (Ans. 20,000 dynes.)

   (ii) Show, if the string breaks, the subsequent radial velocity v cm. per sec. of either block, at the instant when its centre of mass is x cm. from the axis, is given by

   $$v^2 = 4(x^2 - 100).$$

4. If, in the apparatus of question 3, a stop is placed in each arm of the horizontal rod, so that the blocks are brought to rest when their centres of mass are each 26 cm. from the axis, calculate

   (i) The radial velocity of each block immediately before impact;

   (Ans. 48 cm. per sec.)

   (ii) The *impulse** of the blow inflicted by each block on its stop.

   (Ans. 24,000 dyne-secs.)

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* For the definition of the *impulse* of a force, see Appendix I., § 16.

* EXPERIMENT XXXVII.

RADIUS OF GYRATION OF A FLY-WHEEL.

Object of Experiment:

To show from experimental data that the radius of gyration † k of a uniform solid wheel of radius a about its axis is given by the relation ‡

$$k^2 = \frac{1}{2}a^2$$

Apparatus:

Heavy solid cylindrical fly-wheel mounted in ball-bearings (fig. 37), ½-lb. and 1-lb. ring-weights, long rule, special paper for wave-tracing, gum, Capstan stand fitted with adjustable platform.

Outline of Method:

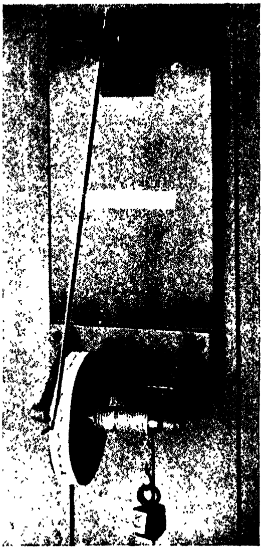

FIG. 37.

A string is wound round the axle of the wheel, and a weight (w lbs. wt.) is attached to it This driving weight is allowed to fall a distance h ft. by first supporting it on a platform fixed at a height of h ft. from the ground, and then removing the platform.

The peripheral velocity of the wheel at the instant when the weight reaches the ground is measured by allowing the spring (fig. 37) to record a wave-tracing on the paper previously wrapped round the circumference of the wheel.

The small retarding force due to the frictional resistance of the wheel-bearings is eliminated by attaching a suitable friction-rider to the string carrying the driving weight.

---

† See Appendix I., § 19.

‡ For a theoretical proof of this result, see Appendix II., Note 6.

SECTION I.

Theoretical.

Let M (lbs.) = mass of wheel,
w (lbs.) = mass of driving weight,
h ft. = distance fallen by driving weight,
a ft. = radius of wheel,
b ft. = „ „ axle,
k ft. = radius of gyration of wheel about its axis,
V ft./sec. = peripheral velocity of wheel at the instant when the driving weight reaches the ground,
u „ = peripheral velocity of the axle at this instant,
U „ = „ „ of the rim of the simple equivalent fly-wheel.

The work done by the driving weight in falling the distance h is used up:

(i) in giving a final velocity V to the wheel,
(ii) „ „ „ u to the driving weight,
(iii) in overcoming the friction of the wheel-bearings (the use of the friction-rider eliminates this factor).

Now, work done by driving weight = wgh foot-poundals
Kinetic energy acquired by wheel = $\frac{1}{2}MU^2$ „
and that acquired by driving weight = $\frac{1}{2}wu^2$ „

Hence, employing the principle of EXPERIMENT XXVI., viz.:—

Change of Kinetic energy = work done,

we have the equation

$$\tfrac{1}{2}MU^2 + \tfrac{1}{2}wu^2 = wgh$$

therefore

$$U^2 = (2wgh - wu^2)/M \quad \ldots\ldots (1)$$

but $U^2/V^2. = k^2/a^2$: and $u/V = b/a$

thus $U^2 = k^2V^2/a^2$: and $u^2 = b^2V^2/a^2$,

hence equation (1) gives

$$k^2/a^2. = 2wgh/V^2M. - wb^2/a^2M \quad \ldots\ldots\ldots\ldots (2)$$

It is, then, the object of the experiment to show that the right-hand side of equation (2) is, within the limits of experimental error, equal to $\frac{1}{2}$.

SECTION II.

Experimental.

(1) *Setting up of apparatus* :—

The instrument may be screwed to the wall, as shown in the figure, or simply laid horizontally on the table.

In the latter case, weights should be placed on the end of the board, remote from the wheel to prevent overbalancing.

The bearings should be oiled before use.

(2) *Preliminary test of friction-rider* :—

Wrap a piece of the special paper round the rim of the flywheel, sticking the two ends together with gum or seccotine.

Attach the friction-rider only to the string wrapped round the axle. This string should be as thin as will support the weight without stretching. A suitable friction-rider may be made by cutting a strip off a piece of sheet brass or copper, its size being adjusted by trial. (If the size has been underestimated, it may be brought up to the required weight by dropping a lump of solder on to it.)

Now take a wave-tracing (having previously drawn a line down the centre of the paper with the stationary brush), setting the wheel in motion by hand.

If the friction-rider is accurate, all the wave lengths so obtained should be equal.

If they are not equal, the size of the friction-rider must be altered until they are.

(3) Now fasten a fresh piece of paper round the rim of the wheel, and draw a line down its centre with the stationary brush, turning the wheel slowly round with the hand.

(4) Attach a $\frac{1}{2}$-lb. ring-weight, together with the correct friction-rider, to the string wrapped round the axle, and place it upon a platform, projecting from a Capstan stand, which has been carefully set at a measured height h ft. above the ground.

(5) Holding the weight carefully at the correct height, remove

the platform, and set the wheel in motion by releasing the driving-weight.

(An alternative method is to dispense with the platform, and have the weight held at the correct distance from the ground by a fine thread. By burning the thread, the weight is released.)

(6) The brush should not be allowed to record on the paper band until the instant when the weight has reached the ground; and then the wheel should be stopped with the hand when it has completed one revolution.

Experimental Results.

The following are the details of an experiment performed with this apparatus :—

a = 0·2915 ft.,
b = 0·04166 ft.,
h = 3·019 ft., (= 98 cm.)
M = 20·1 lbs.,
w = 1 lb.,
g = 32·195 ft./sec.^{2}

Time of vibration of spring 1/6 sec.
Wave-length at instant when w was removed = 8·7898 in. (average of two experiments.)
Thus V = 4·3949 ft. per sec.

Substituting these results in the formula

$$k^2/a^2 = 2wgh/MV^2 - wb^2/Ma^2,$$

we have $$k^2/a^2 = 0{\cdot}5005 - 0{\cdot}0010 = \mathbf{0{\cdot}4995}.$$

Thus, the observed value of k^2/a^2 is equal to $\frac{1}{2}$, correct to 1 part in 1000.

* EXPERIMENT XXXVIII.

THE COMPOUND PENDULUM.

Object of Experiment †:

Preliminary Experiment.

Observations are to be made of the small ‡ oscillations in a vertical plane of a heavy straight rectangular rod of uniform density and cross-section fitted with adjustable knife-edges, in order to determine the manner in which the period of oscillation about a knife-edge varies with the distance of the knife-edge from the centroid of the rod.

Kater Type of Experiment for the Determination of g.

The uniform rod used in the Preliminary Experiment is loaded at one end with an additional weight sufficient to bring the centroid well away from the middle of the rod. One knife-edge is fixed near this weight, while the other is moved about. By trial a position of the movable knife-edge is found for which the periods about either knife-edge are approximately the same.

Two further pairs of observations lead to a graphical method by which the correct position of the movable knife-edge for exact equality of periods is calculated. This common period T_0 is then that of a simple pendulum whose length is the distance L_0 between the knife-edges.

The truth of this is tested experimentally; and the value of g is calculated from the formula

$$T_0 = 2\pi \sqrt{L_0/g}.$$

Apparatus Required for the Preliminary Experiment:

Straight metal rod of uniform rectangular cross-section graduated throughout its length and fitted with two adjustable opposed

---

† Before proceeding with this experiment, the student should read Note 10 of Appendix II.

‡ Say, not exceeding 4° on either side of the vertical. See Appendix II., Note 3.

knife-edges. (The rod used in the experiment described below was of brass, length 66 in., graduated from 0 to 65 in., with a $\frac{1}{2}$ in. blank at each end; its cross-section was a square of side $\frac{1}{2}$ in., and its weight 5 lbs.)

Suspension bracket clamped to the wall at a height of about 6 ft. (*see* fig. 38*a*).

Stop-watch, or clock with seconds pendulum. (An electric chronograph driven by such a clock is still more convenient.)

Parallel-jaw calipers; set-square, or small carpenter's square.

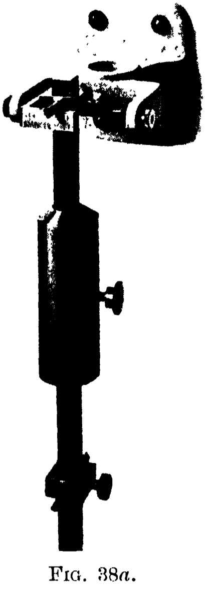

FIG. 38*a*.

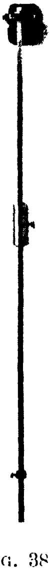

FIG. 38*b*.

Apparatus Required for the Kater Experiment:

The rectangular rod used in the Preliminary Experiment, together with a loading weight (shown in figs. 38*a*, 38*b*) *to be clamped to one end so as to bring the centroid of the whole away from the middle point of the rod. (The loading weight used in the experiment described below was* 3·9 *lbs., and when fixed right* *at the bottom of the rod brought the centroid from the 32·5 in. to the 46·5 in. graduation.*)

The same time-measuring device as used in the Preliminary Experiment.

Spherical plummet (steel or brass) and thread to form "simple pendulum."

Historical Note.*

One of the earliest determinations of the value of g was made by Huyghens at Paris in 1673. Using a "simple" pendulum—that is, a pendulum of the type employed in Experiment XXX., consisting of a plummet suspended by a fine string or wire—he obtained $g = 32{\cdot}16$ ft./sec.^{2}

Such a pendulum is, however, only an approximation to a true simple pendulum, which presupposes ideal conditions incapable of actual realisation. The experimental pendulum must have a bob of finite size suspended from a string of finite mass; and if great accuracy is required, corrections must be made for both of these factors, as well as for changes in the length of the string due to variations of temperature, and for the buoyancy of the displaced air.

The difficulty of making these corrections suggested search for an alternative method, and led to the development of the theory of the compound, or solid pendulum. Huyghens proved that if parallel axes on a compound pendulum are chosen such that the period of oscillation about either of them is the same, while the centroid lies in the plane of the two axes, between them, and at unequal distances from them, then *the common period is the same as that of a simple pendulum whose length is equal to the distance between the axes.* (A theoretical proof of this principle is given in Appendix II., Note 10.) It is necessary to stipulate that the centroid must not be midway between the axes; for if it were, the periods about both axes would be equal whatever their distance from the centroid.

This property of a compound pendulum was first made use of in a gravity determination by Capt. Kater in 1817. His pendulum consisted of a straight rod carrying two adjustable weights, one large and the other small. There were two fixed knife-edges facing inwards on opposite sides of the centroid, so that the pendulum could be swung from either. The larger weight was moved about until the periods of oscillation about either knife-edge were nearly equal; the smaller weight was then moved by a fine screw until exact equality of periods was obtained. Then, in accordance with the above principle discovered by Huyghens, the length of the equivalent simple pendulum was obtained by careful measurement of the distance between the knife-edges.

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* This historical note is mainly based on Poynting and Thomson's *Properties of Matter* (Griffin & Co.), to which the student is referred for a detailed account of methods of determining g.

In this classical determination, elaborate precautions were taken to secure accuracy. The period of oscillation was determined by the method of coincidences. A white circle on black paper was fastened to the pendulum bob of the standard clock employed. Kater's pendulum was suspended in front of the clock, so that when the two pendulums were at rest the tail-piece of the former just covered the white circle on the latter as viewed in a telescope placed a few feet away. A coincidence was the instant at which the white circle was invisible in the telescope as, during an observation, the two pendulums swung past the lowest point together. A series of swings was taken, first from one knife-edge and then from the other, each series lasting about 80 minutes. After each series the small weight was moved until the number of vibrations per 24 hours only differed by a small fraction of one vibration whichever knife-edge was used. The clock was standardised from astronomical observations.

Corrections were made for finite amplitude, for variations of temperature and air-pressure, as well as for the buoyancy of the displaced air. Other corrections necessary were for the "damping" action of air-friction (see Appendix II., Note 2*a*), for yield of support, for elastic lengthening and bending of the pendulum, and for the rolling and slipping of the knife-edge.

The final value of the length of the seconds-pendulum, after reduction to sea-level, in the latitude of London was taken to be 39·13929 inches.

Discussion of Preliminary Experiment.

Before using a pendulum of the Kater type, the student will find it very instructive to examine the behaviour of a perfectly symmetrical pendulum consisting of a straight rod of rectangular cross-section (the rod of fig. 38*a* with the loading weight removed), fitted with two adjustable opposed knife-edges. The pendulum is, in fact, a rectangular parallelopiped, a circumstance which renders easy a comparison between theory and experiment.

Let T = period about knife-edge distant h from centroid (centre) of rod,
K = radius of gyration* of rod about axis through its centroid perpendicular to its length.

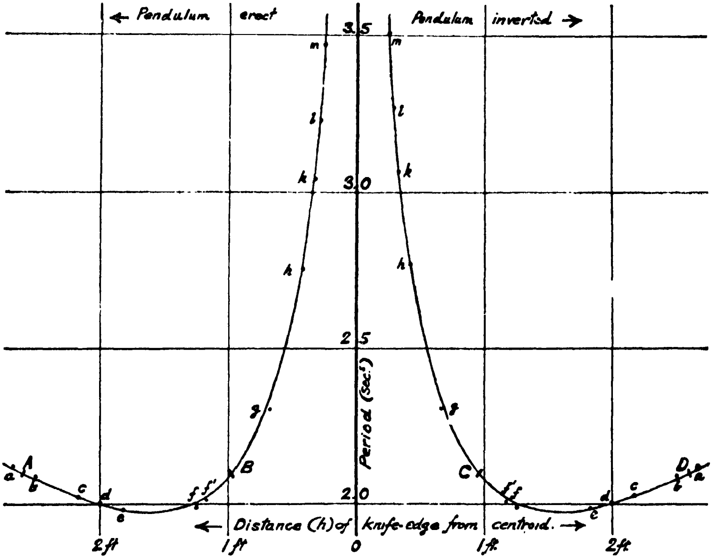


FIG. 38c.—Relation between h and T.

Then it is proved in Note 10 of Appendix II., that T is given by

$$gT^2 = 4\pi^2(h + K^2/h).$$

The graph of this relation between h and T is given in fig. 38c.

---

* The definition of *radius of gyration* is given in Appendix I., § 19, while it is proved in Appendix II., Note 7, that for this rod K is given by

$$K^2 = (a^2 + b^2)/3,$$

where $2a$ is the length, and $2b$ the side of the square cross-section.

Deductions from the Curve.

(i) Within a certain range, there are four points on the rod, two on either side of the centroid, such as A, B, C, D, about which the period is the same.

The points A and D, B and C, respectively, are equidistant from the centroid.

(ii) It may be proved, that, if either unsymmetrical pair AC or BD be chosen, then a simple pendulum of length AC (or BD) has the same period as the common period about the four points. (See Appendix II., Note 10.)

(iii) Outside the above range, T increases continuously as the knife-edge approaches the centroid; the T-axis is an asymptote* to the curve, indicating that the period of oscillation about the centroid is, theoretically, infinite.

(iv) When the period is a minimum, the distance of the supporting knife-edge from the centroid is equal to the radius of gyration K about the centroid. (The proof of this is left as an exercise to the student.)

Elaborate precautions and the expenditure of a good many hours (the timing of a large number of swings for each position) are required to obtain experimentally points lying exactly upon this curve. The student who has not this time at his disposal will, however, find it well worth his while to see how near he can get to this theoretical curve with comparatively rapid observations (say, the timing of 20 swings for each position) carried out over the whole length of the rod.

The procedure recommended in carrying out this experiment is now given, together with a few specimen readings, which are also plotted on fig. 38c.

Procedure.

(1) Ascertain that the centroid is accurately at the middle point of the rod by suspending it horizontally from this point by a piece of string. In the specimen rod the centroid was at the graduation 32·5 in.

(2) Set one knife-edge so that its edge is exactly at the graduation 0·5; so that $h = 32$ in.

As the rectangular sleeve from which the knife-edge projects prevents this setting being made directly, the adjustment may be made in the following way:—

---

* That is, a tangent whose point of contact is infinitely distant.

Using the side of the calipers intended for internal measurements, open them to $\frac{1}{2}$ in.; place the set-square at the graduation 1·0, and then slide the knife-edge until the gap between it and the edge of the square, as shown by the calipers, is $\frac{1}{2}$ in.

(3) In order that the additional weight of the knife-edge and its attachments may not disturb the position of the centroid, set up the other knife-edge, by a similar process, at the corresponding position at the other end of the rod, viz. at the reading 64·5 in.

(4) In order to make sure that the oscillations are within the prescribed limits, hang up on the wall exactly behind the pendulum a wooden rod to which is attached, transversely, a scale indicating the amplitudes corresponding to angular deflections of 2°, $2\frac{1}{2}$°, 3°, on either side of the vertical. This rod also serves as a Mean position mark, and should therefore be of the same width as the pendulum.

(5) Set the pendulum on the bracket in the erect position, i.e. hanging from the bracket at the 0·5 graduation.

(6) Set the pendulum oscillating through exactly 3°. A convenient way to secure this is to set up a Capstan stand with a rod projecting horizontally from it. Move the stand until the rod is against the 3° mark on the scale, and carefully at right angles to the plane in which the pendulum will swing. Now take a short length of thread and loop it round the pendulum; then pull the pendulum aside with the thread until it is in contact with the projecting rod. If the thread be now dropped the pendulum is set in motion in a vertical plane with initial amplitude 3°.

(7) Observe the time of twenty transits of the Mean position, say, from left to right. (The observer counts to himself "Nought," "One," "Two," etc.) Hence determine the period T about this point.

(8) Invert the pendulum, and similarly obtain the corresponding period T' about the equidistant point.

(9) The operations (2) to (8) must now be systematically repeated for values of h decreasing continuously in steps of 1 in.

A few specimen readings are now given, selected from a set of actual observations on a rectangular brass pendulum of the dimensions stated :—

TABLE XXXVIIIa.

Observations on Rectangular Pendulum.

Observation Number.	h (in.).	T (erect) sec.	T' (inverted) sec.
a........................	32	2·123	2·125
b........................	30	2·095	2·095
c........................	26	2·032	2·031
d........................	24	2·009	2·007
e........................	22	1·986	1·988
f........................	15	1·989	1·988
f'........................	14	2·015	2·007
g........................	8	2·302	2·302
h........................	5	2·759	2·771
k........................	4	3·041	3·063
l........................	3·5	3·237	3·274
m........................	3	3·476	3·507
n........................	2	4·302	4·295
p........................	1	5·962	5·962
q........................	0·5	8·531	9·052

The observations *a* to *m* are plotted in fig. 38*c*. The values of T and T' in the above table were obtained by timing twenty swings with an electric chronograph driven by a clock beating seconds. The last figure in each case must be regarded as an estimated figure only. The whole set gives an idea of the degree of accuracy obtainable with comparatively rapid observations of this kind. It will be noticed that there is a fair degree of equality between T and T' so long as h is large; when, however, the knife-edge is very close to the centroid, the instrument, when hanging freely, is very nearly in neutral equilibrium; a circumstance which makes considerable demands on the close conformity of the pendulum to the ideal geometrical shape presupposed.

In order to see how accurately the Deduction (ii) on p. 174 is borne out by these experiments, we may select either unsymmetrical pair of *d* and *f'*, for which the periods are very nearly equal (average = 2·01 sec.)

From the Table, length of equivalent simple pendulum

$$= 24 + 14 = \mathbf{38\ in.}$$

The graph, however, shows that for this period the length should be

$$2{\cdot}07 + 1{\cdot}22 \text{ ft.},$$
$$= 3{\cdot}29 \text{ ft.} = \mathbf{39{\cdot}5\ in.}$$

The observed value thus differs from the theoretical by about 3·8 per cent.

Modified Kater Experiment for the Determination of *g*.

A perusal of the Historical Note above will show that Kater's method of having fixed knife-edges and securing equality of periods about them by adjustment of a sliding weight demands a great expenditure of time.

A more rapid way of securing the desired result with a considerable degree of accuracy is to have the loading weight and one knife-edge F fixed, and to move the other knife-edge M. In this case it is not necessary actually to attain the "proper" position M_0 of M for exact equality. Instead we proceed as follows:—

Find by trial two positions M_1 and M_2 of M which lie on opposite sides of M_0; then plot on a graph as ordinates the periods about M_1 and M_2, as well as the corresponding ones about the knife-edge F, the abscissæ being the distances between the opposed knife-edges. These four points may be called M_1, M_2, F_1, F_2 respectively (fig. 38*d*).

We now assume, as a first approximation, that the paths between M_1 and M_2, F_1 and F_2 are straight lines. Join these points, then, by straight lines, and let M_0 be their point of intersection. The abscissa and ordinate of M_0 should then be the proper distance between the knife-edges, and the common period.

It is to be observed, however, that this construction only approaches accuracy when the distance between M_1 and M_2 is very small; for in reality *the path between M_1 and M_2 is not a straight line, but is a portion of a curve of the type of fig.* 38*c*, *necessarily modified by the asymmetry introduced by the loading weight.*

The same consideration applies, though to a much smaller extent, to the path between F_1 and F_2.

Even with a separation of several inches, however, the construction will indicate a comparatively narrow region within which M_0 must lie; and it is suggested that the quickest way of getting an accurate M_0 is to apply the construction twice over, first with a separation of several inches (fig. 38*d*), and then with a smaller separation (fig. 38*e*), the range for which is deduced from the former figure. The procedure will be made clearer by an actual example:—

Experiment performed with Kater type pendulum composed of a brass rod weight 5 lbs., graduated 0—65 in., loaded with weight 3·9 lbs. at 65 end. Fixed knife-edge at 60 in. graduation.

Preliminary timing observations were made with a stop-watch, taking only 10 swings.

The period about F was found to be but slightly affected by the position of M, and was in the neighbourhood of

$$T_F = 2{\cdot}15 \text{ sec.}$$

When M was at the zero graduation, the period about it was

$$T_M = 2{\cdot}39 \text{ sec.}$$

M was then moved further down the rod. After a few more observations, it was evident that the graduations 14 and 17 were on opposite sides of the proper position M_0.

More accurate observations were now made with M set carefully at each of the above positions; the periods being derived from the timing of 20 complete oscillations with an electric chronograph driven by an Astronomical Clock beating Mean Solar seconds.* (To secure equal accuracy with a stop-watch, at least twice as many oscillations should be taken.)

The results are set forth in the following table:—

TABLE XXXVIIIb.

Preliminary Observations for "Proper" Position M_0.

Movable Knife-edge M at Graduation.	Distance L between Knife-edges.	Period T_M about Movable Knife-edge.	Period T_F about Fixed Knife-edge.
in.	in.	sec.	sec.
14	46	2·1569	2·1448
17	43	2·1293	2·1455

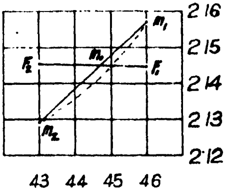
---

* The clock was at the time losing about 6 seconds per day, or 0·00007 sec. per sec.; an error which, to the degree of accuracy considered here, may be disregarded.

These results are plotted on the graph of fig. 38*d*. Proceeding as though the path between M_1 and M_2 were a straight line, we deduce

Proper distance between knife-edges = 44·7 in.
Common period = 2·145 sec.

The correct path between M_1 and M_2, however, is not straight; as already stated. Its exact shape is difficult of calculation, but we have reason to suppose that it is concave to the *T*-axis—somewhat resembling the dotted line of fig. 38*d*. This suggested trying L = 45 in. as the lower limit for more accurate computation of M_0.



First observations for M_0

FIG. 38*d*.

The following observations were then obtained:—

TABLE XXXVIIIc.

Second Observations for "Proper" Position M_0.

M at Graduation	L.	T_M.	T_F.
in.	in.	sec.	sec.
15	45	2·1442	2·1529
14·5	45·5	2·1533	2·1500

These results are plotted on the graph of fig. 38*e*., and the same linear construction as before gives:—

Proper distance between knife-edges L_0 = 45·38 in.
Common period about either knife-edge T_0 = 2·1507 sec.,*

which is to say that a simple pendulum of length 45·38 inches should have a period of 2·1507 seconds.

In order to test this experimentally, a "simple pendulum," consisting of a steel sphere of diameter 1·25 in. suspended from a piece of thin string, was set up, the top of the string being held in a small clamp. The distance from the top of the string to

---

*** Since L_0 is only determined to 4 figures, we are not justified in retaining more than 4 figures in the value for T_0; we accordingly take T_0 = 2·151 sec., which gives g = 32·275, i.e., 32·28 to the same degree of accuracy.**

the centre of the sphere was adjusted as carefully as possible to be 45·38 in. 20 oscillations were timed with the chronograph, and the result gave :—

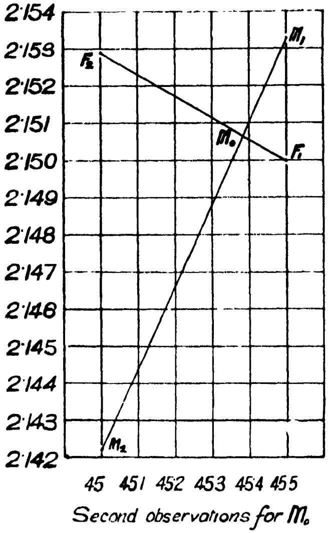


FIG. 38*e*.

Period of experimental "simple pendulum" = 2·1513 sec.
Difference from T_0 = 0·004 per cent.

We may regard this, then, as an experimental proof of the principle (due to Huyghens) deduced theoretically in Note 10 of Appendix II., viz. :—

The centres of "Oscillation" and "Suspension" of a compound pendulum are convertible; and the distance between them is equal to the length of the Simple pendulum whose period is the common period about these centres.

The value of g derived from this experiment is then given by the relation

$$gT_0^2 = 4\pi^2 L_0,$$

giving g = **32·28 ft./sec.,^{2}** correct to 4 figures.

Experimental Exercises.

1. Calculate the radius of gyration K about the axis through the centroid perpendicular to its length for the rectangular rod used in the PRELIMINARY EXPERIMENT, from the formula

$$K^2 = (a^2 + b^2)/3,$$

given that $a = 33\text{in.}, \quad b = \frac{1}{4}\text{ in.}$

(Ans. K = 19·05 in.)

2. Verify from the graph of fig. 38*c* that when the period of oscillation is a minimum, the distance of the supporting knife-edge from the centroid is equal to the above radius of gyration K.

If there is any divergence, express it as a percentage of the calculated value.

Give a theoretical proof of this principle.

3. From the relation $gT'^2 = 4\pi^2(h + K^2/h)$, calculate T' for the values $h = 2, 1, \frac{1}{2}$ in., using the value of K calculated in Exercise 1.

   Compare these values of T with the experimental readings n, p, q of Table XXXVIII*a*.

   To what cause do you attribute differences between corresponding values of T and T' in this Table?

4. Calculate the value of g which Kater's value for the length of the seconds-pendulum at London would give.

   (Ans. $g = 32{\cdot}1908$.)

5. Taking the value of g for Shrewsbury, as obtained from Helmert's formula (see Appendix I., § 20), as the correct value for the latitude of the Laboratory, calculate the percentage error in the value 32·28 obtained in the above Experiment.

   (Ans. + 0·28 per cent.)

APPENDIX I.

DEFINITIONS, FORMULÆ, AND SUMMARY OF MAIN PRINCIPLES.

Part I.—STATICS.

§ 1. **Moment of a Force.**

The moment of a force F about a point O $= F \times p$ (fig. 1), where p is the length of the perpendicular from O on the line of action of the force. This moment may be either clockwise or anticlockwise in direction. (Experiments I., II., and III.)

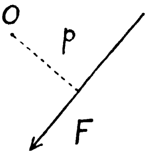


FIG. 1.

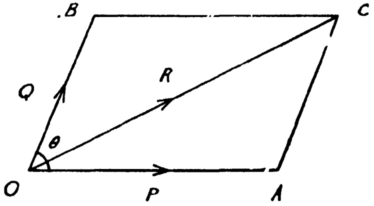


FIG. 2.

§ 2. **Triangle of Forces.**

If three coplanar forces act at a single point of a body in such a way as to maintain it in equilibrium, they may be represented, in magnitude and direction, by the sides of a triangle taken in order. (Experiment VIII.)

§ 3. **Resultant of two or more forces.**

The single force which can, for mechanical purposes, replace these forces.

Parallelogram of Forces.

If two forces P and Q be represented, in magnitude, direction, and sense, by two straight lines OA, OB; the resultant R of P and Q is similarly represented by the diagonal OC of the parallelogram determined by OA and OB (fig. 2).

If θ be the angle between OA and OB, the relation between P, Q, and R is

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta.$$

§ 4. Resolved Part of a force F in a direction inclined at an angle θ to the direction of F.

Draw a straight line OC to represent F in magnitude and direction, and let OX be the required direction (fig. 3); then Resolved Part of F in direction OX

$= \text{projection of OC on OX}$
$= \text{OA}$
$= F \cos \theta.$

The Resolved Part in the direction perpendicular to OX $= \text{projection of OC on OY}$
$= \text{OB}$
$= F \sin \theta.$ (Experiment X.)

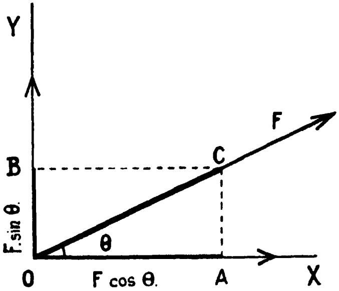


Fig. 3.

§ 5. Couple and Torque.

Two equal parallel forces acting in opposite directions are said to form a **Couple.**

The moment of a couple about any point in its plane is the same quantity, and is called the **Torque** of the couple.

It may be proved that, if F be the magnitude of each force and d the perpendicular distance between them,

$$\textbf{Torque} = F \times d \qquad \text{(Experiment XI.)}$$

§ 6. Coefficient of (sliding) Friction ($= \mu$) of one substance with regard to another.

If F be the force which must be applied parallel to the surface of contact of the two substances in order to make the one substance move slowly over the other, and N be the reaction normal to the surface of contact, then

Coefficient of Friction $\mu = F/N$. (Experiment XII.)

§ 7. Types of Equilibrium.

If a body, in equilibrium under a system of forces, is slightly displaced, then the system of forces will either

(i) tend to restore the body to its former position; or
(ii) tend to make it depart farther from that position; or
(iii) still preserve equilibrium.

In case (i) the equilibrium is called **stable.**
In case (ii) the equilibrium is called **unstable.**
In case (iii) the equilibrium is called **neutral.**

§ 8. Centre of Gravity, or Centroid. (Experiments XIII., XIV.)

That point of a body at which acts the resultant of all the parallel forces due to gravitational attraction on the several particles of the body.

If the body be suspended about an axis through this point, it will remain in (neutral) equilibrium at any orientation.

In cases where the mass, rather than the weight, of the body is under consideration, the point is called the **centre of mass.**

This point is usually denoted by the letter G. Hence the term "the particle G" is a convenient abbreviation for "a particle situated at the centre of gravity (centre of mass) whose weight (mass) is equal to that of the body."

§ 9. Velocity-ratio of a machine for raising a load.

The distance the point of application of the effort moves while the load rises unit distance.

Or, more generally,

The ratio of the distance the point of application of the effort moves to that which the load ascends.

Mechanical Advantage of the machine.

The ratio of the load to the effort which is just sufficient to raise the load ($= L/P$).

Efficiency of the machine.

The ratio of the work done on the load to that done by the effort.

If e be the efficiency, and V the velocity-ratio, then in general

$$e = L/VP.$$

In order that the load may hang stationary when the effort is removed, the efficiency must be less than 50 per cent. (Experiments XV.-XVIII.)

§ 10. **Hooke's Law** (for elastic materials) :—

The extension produced by a given load is proportional to that load.

Strain : The increase in length of unit length.

Thus if a wire of length l becomes extended to l',

$$\text{strain} = (l' - l)/l \text{ ... .} \quad (16)$$

Stress : The load per unit area of cross section.

Thus if the above wire be of cross-section a circle of radius a cm., and W kgs. wt. be the load upon it,

$$\text{stress} = W/\pi a^2 (\text{kgs. wt./cm.}^2) \text{..............} \quad (17)$$

Young's Modulus = **stress/strain.**

$$= Wl/\pi a^2(l' - l) \quad (\text{kgs. wt./cm.}^2) \text{...} \quad (18)$$

(Experiment XIX.)

Elastic Limit : If an elastic material be overloaded, the extension is greater than that given by Hooke's Law, and the material fails to regain its unstretched length on removal of the load. The material is said to have reached its **elastic limit** when the extension begins to depart from Hooke's Law.

Yield-point : A point reached by overloading at which the material draws out excessively without further increase of load. A small additional load produces rupture. (Experiment XX. See also Appendix II., Note 1).

Part II.—DYNAMICS.

§ 11. **Symbols employed.**

s = Distance (centimetres or feet) after time t.
m = Mass (grammes or pounds).
w = Weight (grammes' weight or pounds' weight).
t, T = Time (seconds or vibrations).

u, v = Initial velocity, and velocity after time t, respectively. The symbol v_1 is also sometimes employed for initial velocity.

The velocity of a moving point may be defined as its *rate of change of position* (cm./sec., or ft./sec.).

U = Average velocity = (total distance)/(total time).
a = Acceleration = increase of velocity in 1 second (or in unit time).
$-a$ = Retardation = decrease of velocity in 1 second (or in unit time).
g = Acceleration of gravity (cm./sec.^{2} or ft./sec.^{2}), which may be experimentally proved to be the same for all bodies independently of their masses.
F = Force (dynes or poundals).

1 dyne = force required to give an acceleration of 1 cm. per sec. per sec. to a mass of 1 gm.
It is roughly equal to the weight of a milligramme.

1 poundal = force required to give an acceleration of 1 ft. per sec. per sec. to a mass of 1 lb.
It is roughly equal to the weight of half an ounce.

W = Work = Force × distance (ergs or foot-poundals).

Other quantities employed.

Momentum = mass × velocity = mv.

Kinetic Energy = half mass × square of velocity = $\frac{1}{2}mv^2$.

Power = rate of work = work done per second (or per unit time). (e.g. ergs per second or foot-poundals per second.)

1 **Horse-Power** = 550 ft.-lbs. per sec. = 33,000 ft.-lbs. per minute.

1 **Watt** = ten million ergs per second = 1 volt × 1 ampere (numerically).

1 **Kilowatt** = 1000 watts = $1\frac{1}{3}$ horse-power (approximately). Also (more accurately)
746 watts = 1 horse-power.

§ 12. **Newton's Laws of Motion.**

Law I. Every body tends to remain in its state of rest, or of uniform motion in a straight line, unless it be compelled by some external impressed force to change that state.

Law II. The rate of change of momentum produced in a body by an impressed force is directly proportional to

that force, and takes place in the direction in which the force acts.

Law III. To every action there is an equal and opposite reaction.

§ 13. Experimental Laws.

Law I. A constant force acting upon a body produces in it a constant acceleration. (Experiment XXII.)

Law II. If the force acting upon a body be varied, the acceleration produced is directly proportional to the magnitude of the force. (Experiment XXII.)

Law III. If the same force act upon different bodies, the acceleration it produces is inversely proportional to the mass of the body. (Experiment XXIII.)

Law IV. For motion under a constant acceleration, the average velocity is half the sum of the initial and final velocities. (Experiment XXIV.)

$$\text{i.e. } U = \tfrac{1}{2}(v_1 + v_2) \ldots\ldots \quad (1)$$

Mathematical expression of Laws I., II., and III.

If the unit of force be correctly chosen, the above three Laws express the fact that

The ratio of the force to the acceleration is numerically equal to the magnitude of the mass acted upon;

$$\text{i.e. } F = ma \ldots\ldots \quad (2)$$

(The units of force referred to in § 11 are chosen in this way.)

Thus these three experimental laws are particular aspects of Newton's Second Law.

§ 14. Relation between mass m and weight w.

In view of the fact that it has been ascertained experimentally that the acceleration g of gravity is the same for all bodies independently of their mass, it follows, in accordance with equation (2), that

$$w = mg \ldots\ldots \quad (3)$$

§ 15. Motion with constant velocity v.

Distance described in time t $\quad s = vt \ldots\ldots$ (4)

Motion from rest under constant acceleration a.

Velocity at time t from the start $\quad v = at \ldots\ldots$ (5)

Average velocity $\ldots\ldots$ $U = \tfrac{1}{2}(0 + at)$

$$\text{i.e. } U = \tfrac{1}{2}at \ldots\ldots \quad (6)$$

Distance moved in time t............ $s = U \, . \, t = \frac{1}{2}at \, . \, t$

i.e. $s = \frac{1}{2}at^2$ ......(7)

By eliminating t between (5) and (7), we have an equation giving the velocity acquired by moving from rest a distance s under acceleration a when the time is not known, viz. :—

$v^2 = 2as$.........(8)

Motion with initial velocity u under constant acceleration a.

Velocity at time t from the start $v = u + at$ ....(9)

Average velocity .... $U = \frac{1}{2}(u + u + at)$

i.e. $U = u + \frac{1}{2}at$ ......(10)

Distance moved in time t ......$s = U \, . \, t = (u + \frac{1}{2}at) \, . \, t$

i.e. $s = ut + \frac{1}{2}at^2$......(11)

By eliminating t between (9) and (11) we have an equation giving the velocity acquired by moving a distance s under acceleration a when the time is not known, viz. :—

$v^2 = u^2 + 2as$...... (12)

In the case of motion under a **retarding** force, $-a$ must be substituted for a in the above formulæ.

In the case of vertical motion under gravity, put $a = g$ if the motion is **downwards**, and $a = -g$ if the motion is **upwards.**

In a great many cases it is sufficient to take

$g = 981$ cm./sec.^{2} or 32 ft./sec.^{2}

The correct value of g for a number of places is given in the table in § 20 below.

§ 15*a*. Angular velocity.

If a point P is revolving about another point O, the angular velocity ω of P relative to O is measured by the angle through which P would move in a unit of time (**unit ; 1 radian per second**).

If v be the peripheral[*] velocity of P, and OP $= r$, then

$v = r\omega$.............................. (13).

§ 16. Impulse of a force F.

Measured by $F \, . \, t$ (dyne-seconds),
where t is the time during which the force acts.

---

[*] That is, the velocity along the periphery, or circumference.

In many cases of impact between bodies, it is impossible to calculate the force of the impact; for in order to do so, we need to know the retardations or accelerations produced in the several bodies, and this involves a knowledge of the time during which the force acted, which may be unobtainable.

Thus, for example, if a body of mass m moving with velocity v encounters a stationary object, and is brought to rest, we cannot, unless we know the time of duration of the impact, do more than measure the **quantity of momentum** destroyed, which is

$$mv.$$

If a be the average retardation produced in the short, but unmeasured, time t, then

$$v = at,$$

and so $$mv = mat = Ft.$$

Thus the **impulse of a force is measured by the change of momentum which it produces.**

§ 17. **Period of oscillation T of a Simple Pendulum.** (Experiment XXX.)

Ideally, a Simple Pendulum consists of a particle (whose mass does not enter into the calculation) suspended by a weightless inextensible string of length L (cm. or ft.)

If the angle through which it oscillates about the vertical be indefinitely small, the period T of a complete oscillation is given by

$$T = 2\pi\sqrt{L/g} \dots\dots\dots\dots\dots\dots (14)$$

if the pendulum be suspended *in vacuo* (see Appendix II., Note 3).

As no actual pendulum is more than an approach to this ideal, corrections must be made, for refinements of accuracy, for the following factors, each of which increases the period :—

(i) **Finite angular displacement.** For example, if the angular displacement be 4°, the period is increased by 0·05 per cent. (Appendix II., Note 3).

(ii) **"Damping" action of air-friction.** This point is discussed in Appendix II., Note 2A. The correction is even smaller than that for finite angle.

(iii) **Moment of Inertia of the bob.** If the bob is a heavy sphere of radius a (cm. or ft.) it may be shown (see Appendix II., Note 10, Ex. 3) that the period becomes increased to T', where

$$T' = 2\pi\sqrt{(L/g)(1 + 2a^2/5L^2)} \qquad \ldots\ldots\ldots\ldots\ldots(15)$$

The order of magnitude of this correction is illustrated by Ex. 4 of this Note.

The experiments described in this book, however, do not demand such refinements of accuracy; the student may, therefore, take it as a convenient practical rule that formula (14) will meet his requirements, provided that the initial angular displacement of his experimental "Simple Pendulum" is of the order of 4°.

§ 18. **Centrifugal Force.** (Experiment XXXVI.)

A body, of mass m gms., is free to slide along a smooth horizontal rod OAB (fig. 4).

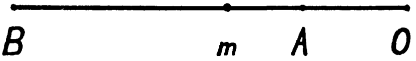


FIG. 4.

If the rod be rotated about O in a horizontal plane, the tendency of m to continue travelling in the straight line in which it started (Newton's First Law), causes it to move outwards from the centre along AB.

An observer fixed with regard to AB would interpret this effect as due to a "centrifugal force," i.e. a force acting on m which tends to make it depart further from the centre O.

At the instant when m is revolving about O with peripheral velocity v (cm./sec.) and at a distance R cm. from it, the magnitude of this centrifugal force is

$$mv^2/R \quad \textbf{(dynes)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

and its direction is AB.

Thus, for example, if a stop were placed at B, m would press against it with a force of

$$mv^2/x \quad \text{(dynes)},$$

where v is m's velocity at that instant, and x its distance from O.

Moreover, the stop would exert an equal and opposite reaction (Newton's Third Law) or *centripetal constraint* upon m.

It will thus be seen that the fundamental cause of centrifugal force is the inertia property of matter referred to in Newton's First Law.

The student must not, however, suppose that if at any instant the constraint upon m's motion were completely removed, it would then fly out from the centre O ; on the contrary, it would move with uniform velocity along the tangent to its path at that instant.

§ 19. Radius of Gyration, and Moment of Inertia.

In many problems on rotating bodies, the calculations are simplified by substituting for the actual body an imaginary wheel—of specially chosen radius k—consisting only of a rim * (of no thickness), around which is uniformly distributed a mass equal to that of the actual body ; the axis of rotation being the axis of the wheel.

The radius (k) of this imaginary " rim-wheel " is called the **radius of gyration** of the body about this axis.

If M be the mass of the body, the quantity

$$I = M \cdot k^2 \quad \text{..........................(17)}$$

is called the **Moment of Inertia** † of the body about the axis of rotation.

k must be such, for example, that the actual body and its " rim-substitute " have the same Kinetic Energy when rotating about the axis with the same angular velocity.

A method of calculating k for a rectangular rod, a solid cylinder, and a sphere, is given in Notes 7 and 6 of Appendix II., while a practical method of determining it for a wheel is given in Experiment XXXVII. (See also Experiment XXVIII., Exercise.)

---

* It is to be observed that a single particle of the same mass placed at any point on this rim would serve equally well, if it is not essential that the centre of mass should be on the axis of rotation.

† Another aspect of the moment of inertia is seen by regarding the body as an assemblage of particles. From this point of view, the moment of inertia is the quantity

$$\Sigma mr^2,$$

where m is the mass of a particle, and r its distance from the axis of rotation, while the Σ denotes summation for all the particles of the body.

The values of k in a few instances are collected below :—

Shape of Body.	Axis of Rotation.	k^2.
Uniform rod, length L (of negligible thickness).	At one end, perpendicular to length.	$L^2/3$.
Thick rectangular rod, length $2a$, depth $2b$. (The width parallel to the axis does not affect the value of k.)	Through centroid, perpendicular to length and depth.	$(a^2+b^2)/3$
Solid cylinder, radius a. (The width parallel to the axis does not affect the value of k.)	Axis of cylinder.	$\frac{1}{2}a^2$
Sphere, radius a.	A diameter.	$2a^2/5$

§ 20. Variation of g with latitude.

The following table gives the values of g, the acceleration of gravity, at the places mentioned. These values are based on the larger table given in Kaye and Laby's *Physical and Chemical Constants* (4th Edition).

Place.	Latitude.	g (cm./sec.^{2}).	g (ft./sec.^{2}).
	° ′ ″		
North Pole	90 0 0 N.	983·216	32·258
Aberdeen	57 8 58 N.	981·680	32·208
Edinburgh	55 57 24 N.	981·584	32·205
Liverpool	53 24 19 N.	981·350	32·197
Shrewsbury	52 42 14 N.	981·281	32·195
Greenwich	52 28 38 N.	981·188	32·192
Paris	48 50 11 N.	980·943	32·184
Rome	41 54 0 N.	980·347	32·164
Calcutta	22 32 48 N.	978·816	32·114
Equator	0 0 0	978·030	32·088
Cape Town	33 56 0 S.	979·659	32·142
Wellington (N.Z.)	41 17 4 S.	980·292	32·162
South Pole	90 0 0 S.	983·216	32·258

The value of g for any given latitude may be calculated from Helmert's formula :—

$$g = 980{\cdot}616 - 2{\cdot}5928 \cos 2\lambda + 0{\cdot}0069 \cos^2 2\lambda - 0{\cdot}0003086H,$$

where λ = Latitude,
H = Height above Sea-level in metres.

APPENDIX II.

MATHEMATICAL NOTES.

Note 1. Hooke's Law, Young's Modulus, and the Breaking-stress of a wire. (See Experiments XIX., XX.)

Hooke's Law: Careful observations show that, if a wire is stretched by a continually increasing load, the increase in length, or extension, produced in it is—for a certain range of loads—*directly proportional* to the load.

Further, provided the extension is not excessive, it is also found that, as the load is reduced, the wire regains at each stage the length which it had previously for that load; and finally, when all the load is removed, it regains its unstretched length.

The Law

Extension is proportional to load

was first formulated * by Robert Hooke in 1676, and is known as Hooke's Law.

Young's Modulus: Hooke's Law is inadequate for furnishing an absolute measure of the elasticity of a given material—independent of the particular specimen employed; for it takes no account of the fact that the extension produced in a wire by a given load depends not only on the load, but also on the following factors: —

(i) **The length of the wire**: Given two wires of the same material and thickness, but of different lengths, the longer one will be extended more than the shorter by a given load.

(ii) **The thickness of the wire**: Given two wires of the same material and of the same length, but of different thicknesses, the thicker one will be extended less than the thinner one by a given load.

Hooke's Law was modified by Young in order to take account of these factors; and may now be stated as follows :—

---

* Though not in these exact words.

Extension of unit length is proportional to load per unit area of cross-section.

This leads to the following definitions :—

(*a*) **Strain**: The extension of unit length of a wire is called the *strain* of the wire.

If the wire be of length l, and be stretched to length l', then

$$\begin{aligned}\text{strain} &= (\text{New length} - \text{old length})/(\text{old length})\\ &= (l' - l)/l.\end{aligned}$$

(*b*) **Stress**: The load per unit area of cross-section of the wire is called the *stress* in the wire.

If the diameter be d, and the load M, the cross-sectional area (if the wire be circular) is

$$\tfrac{1}{4}\pi d^2,$$

and the stress $= M/\tfrac{1}{4}\pi a^2 = 4M/\pi d^2$.

Stress thus measures the *degree of concentration* of force.

Young's emendation of Hooke's Law may therefore now be stated as follows :—

Strain is proportional to stress.

The fraction *stress/strain* is thus (provided the wire is not overstrained) a constant quantity for a given material and independent of its length and thickness.

This fraction is called **Young's Modulus.**

Expression for Young's Modulus ($=Y$) in terms of experimentally determinate quantities :—

If a wire of natural length l cm. and diameter d cm. be stretched to a length l' cm. by a load of M kgs. wt., then from (*a*) and (*b*) above, we have

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{4M/\pi d^2}{(l' - l)/l} = \frac{4Ml}{\pi d^2(l' - l)}(\text{kgs. wt./cm}^2.),$$

or, if the strain is denoted by e,

$$Y = 4M/\pi e d^2 \text{ (kgs. wt./cm.}^2\text{)}.$$

In the case of an elastic spring or string, it is often convenient to express the actual force F extending it (rather than the stress) in terms of the extension x it produces. If a be the natural length of the spring or string, the strain produced by the force F is x/a, and we may write

$$F = k \, . \, x/a,$$

k may be called the *elastic modulus* of the spring or string.

Over-strain of a wire—Yield Point.

If the wire be strained beyond a certain limit—called the **elastic limit**—it is found that it fails to regain its unstretched length on removal of the load; it is then said to have acquired *permanent set.*

Thus, if the wire be strained beyond the elastic limit by a steadily increasing force, the ratio of strain to stress is no longer found to be a constant but increases with increasing load. In the case of plastic materials, such as wrought iron and mild steel, a further change takes place when the load is sufficiently great. A point called the **Yield Point** is reached at which the wire draws out suddenly a considerable length, this extension being generally greater than the whole amount of the extension produced by the smaller loads.

After the Yield Point is reached, the wire continues to extend under increased loads until it is about to break. At that stage the part where the rupture is about to occur draws out much more rapidly than the rest of the specimen, and the cross-section there becomes reduced.

(These properties of a wire are investigated in Experiment XX.)

Note 2. Simple Harmonic Motion.

Def.—A particle is said to describe Simple Harmonic Motion about a point O of its path * if it moves so that at every instant of the motion its acceleration ($\ddot{x}$) along the path is proportional to its displacement x from O, and is directed towards O: i.e. if

$$\ddot{x} = -n^2 x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

---

* The path need not necessarily be rectilinear, but in all cases both the acceleration and the displacement must be measured along the path.

It may be proved that the **period** T* of the oscillation is given by

$$T = 2\pi/n \qquad (2)$$

For a rigorous proof of this the student is referred to a Text Book of Theoretical Mechanics (e.g. Lamb's *Dynamics*, Cambridge University Press): in the case where the path is rectilinear, however, equation (2) may be derived by a simple geometrical construction:—

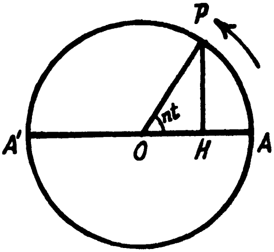


FIG. 1.

Let P be a point moving round the circle APA′ with uniform angular velocity n (radians per second) in the direction of the arrow (fig. 1).

We shall now show that the motion of H, the projection of P on the diameter AA′ satisfies the differential equation (1), i.e. that the motion of H is Simple Harmonic.

Let OA = a, OH = x.

If P starts from A, then at time t we have

$$x = a \cos nt \qquad (3)$$

differentiating with regard to t,

$$\dot{x} = -an \sin nt \qquad (4)$$

differentiating again with regard to t,

$$\ddot{x} = -an^2 \cos nt,$$

$$\text{i.e. } \ddot{x} = -n^2 x \qquad (1)$$

in virtue of (3).

Thus H performs S.H.M. about O.

To calculate the period T of this S.H.M., we notice that it is evidently the same as the time taken by P to travel once round the circle, i.e.

---

* The period of the oscillation is the interval of time which elapses between two consecutive occasions on which H passes a given point of its path *in the same direction*.

The maximum displacement (a) on either side of the mean position is called the **amplitude** of the motion.

$$T = 2\pi/n.$$

Exercises on Simple Harmonic Motion.

Ex. 1. A particle executes Simple Harmonic Motion about a point O. If the amplitude of the motion is 2 ft., and the time of a complete oscillation 2 seconds, write down equations giving its displacement, velocity, and acceleration at time t.

Find also its maximum velocity, and its acceleration when at a distance of 1 foot from O.

(Ans. Maximum velocity $= 2\pi$ ft./sec. $= 6{\cdot}28$ ft./sec.
Acceleration when 1 foot from O $= \pi^2$ ft /sec.^{2}
$= 9{\cdot}87$ ft./sec.^{2})

Ex. 2. A particle, performing S.H.M., has a velocity of 5 ft./sec. when its displacement and acceleration are respectively 12 ft. and 12 ft./sec^2. Find the amplitude and period of the motion.

(Ans. 13 ft., $6{\cdot}28$ ($= 2\pi$) seconds.)

Ex. 3. A particle, performing S.H.M., has a velocity of $5\pi\sqrt{3}$ cm./sec. when its displacement is half the amplitude. If at the same instant the acceleration is

$$10\pi^2 \text{ cm./sec.}^2,$$

find the amplitude and period of the motion.

(Ans. 5 cm., 1 sec.)

Ex. 4. A particle describes a complete Simple Harmonic oscillation in 4 seconds, and its maximum velocity is 10 ft./sec.

Find the amplitude of the oscillation; and hence find the equation for its displacement x at time t. Find also the maximum acceleration.

(Ans. Amplitude $= 20/\pi = 6{\cdot}37$ ft.
$x = (20/\pi) . \cos \frac{1}{2}\pi t$,
Max. accn. $= 15{\cdot}71$ ft./sec.^{2})

Ex. 5. A particle moves with S.H.M.; if when at distances of 3 and 4 ft. from the centre of its path, its velocities be

4 and 3 ft./sec. respectively, find the period of the motion, and the maximum values of the displacement, velocity, and acceleration.

(Ans. Period = 2π secs.,
Max. displact. = 5 ft.
Max. vel. = 5 ft./sec.,
Max. accn. = 5 ft./sec.^{2})

Ex. 6. A particle, moving with S.H.M., has a velocity of 58 cm./sec. when passing through the centre of its path. If its period is π seconds, find its velocity and acceleration when it has described a distance of 8 cm. from the position in which its velocity is zero.

(Ans. 40 cm./sec., 84 cm./sec.^{2})

Ex. 7. A body of mass 5 gms. oscillates through 2 cm. on each side of its mean position 10 times in a second. If the motion be Simple Harmonic, find the maximum force upon the body.

(Ans. $4000\,\pi^2 = 39{,}478$ dynes.)

Ex. 8. A point P moves in such a way that its projections upon two perpendicular lines describe S.H.M.'s of equal period and amplitude about the point of intersection of the lines.

Prove that P moves in a straight line if the two S.H.M.'s are in the same *phase*; * while if they are out of phase by a quarter-period, P will move in a circle.

In the former case, find the angle of inclination of P's motion to either of the perpendicular lines.

(Ans. 45°.)

Ex. 9. A point P moves in such a way that its projections upon two perpendicular lines OX, OY describe S.H.M.'s of equal period about O; the amplitude along OX being twice that along OY.

Prove that P will move in a straight line if the S.H.M.'s are in the same phase; while if they are out of phase by a quarter-period, P will move in an ellipse.

---

* The term *phase* is used to describe the particular position in its path at which the particle is to be found at a given instant; it is mainly used when comparing with another S.H.M.

In the former case, find the angle of inclination of the line of P's motion to the line OX; and in the latter case find the principal axes of the ellipse.

(Ans. Arctan $\frac{1}{2} = 26° 34'$; the semi-axes of the ellipse are the respective amplitudes along OX, OY.)

Ex. 10. If in fig. 1 of Note 2, the time be measured from the instant when angle AOP $= \epsilon$, show that the displacement x of the point H t seconds later is

$$x = a \cos \{(2\pi t/T) + \epsilon\}.$$

(The angle ϵ is called the *epoch* of H's motion.)

Note 2a. Damped Harmonic Motion.

Few cases of free oscillatory motion met with in actual practice are of the pure Simple Harmonic type represented by equations (1) to (4) of Note 2. In the majority of instances a "damping"* factor is present—e.g. in the case of the motion of a pendulum, the resistance of the air—which modifies the motion.

An oscillation of this kind is called, then, a "damped oscillation." Attention is confined here to a damped oscillation of Simple Harmonic type.

It will be shown that the effect of the damping factor is

(1) **To produce a continuous decrease in the amplitude of the oscillation, ultimately bringing the body to rest.**

(2) **To increase slightly the period of the oscillation.** (This effect is the least marked of the two; in the case of a pendulum swinging in the air, it is almost inappreciable.)

Anything like a complete Mathematical treatment is outside the range of this article,† but the subject is so important that an outline of the proof is given here, together with a numerical example.

We assume that the damping force at any instant is proportional to the velocity at that instant.

Equation (1) is then modified to

$$\ddot{x} + 2f\dot{x} + n^2x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1a)$$

---

* A metaphor from the damping of a fire.

† See, for example, J. W. Mellor's *Higher Mathematics for Students of Physics and Chemistry* (Longmans); or a Treatise on Differential Equations.

In order to obtain a solution of this differential equation, put

$$x = ye^{-ft} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2a)$$

Differentiating with regard to t we have

$$\dot{x} = (\dot{y} - fy)e^{-ft}$$

and $$\ddot{x} = (\ddot{y} - 2f\dot{y} + f^2y)e^{-ft}.$$

Substituting these expressions for $\dot{x}$ and $\ddot{x}$ in (1a), reducing and dividing by e^{-ft}, the equation becomes

$$\ddot{y} + (n^2 - f^2)y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3a)$$

A solution of this has already been seen to be (if $n > f$)

$$y = a \cos qt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4a)$$

$$\text{where } q^2 = n^2 - f^2;$$

hence, finally, we have

$$x = ae^{-ft} \cos qt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

which gives the displacement from the mean position at time t.

By comparison with equations (2) and (3) of Note 2, we see that (5a) represents a quasi-Simple Harmonic Motion of period

$$T = 2\pi/q,$$

whose amplitude constantly decreases.

The ratio of successive amplitudes (on opposite sides) is, at all stages of the motion

$$r = 1/e^{-\frac{1}{2}fT} = e^{\frac{1}{2}fT} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6a)$$

Taking logarithms of both sides, we have

$$\log r = \tfrac{1}{2}fT \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

from which equation the damping factor f may be calculated in any practical example.

On account of its importance in this connection, the quantity $\log_e r$ has been called by Gauss the **logarithmic decrement.**

The **logarithmic decrement** of a damped oscillation is thus **the Naperian logarithm of the ratio of successive amplitudes.**

Numerical Illustration :

In order to illustrate the effect of damping in modifying a pure Simple Harmonic oscillation, let us compare the following :—

(i) An undamped Simple Harmonic oscillation of period 4 secs., and amplitude 1 (fig. 1*a*).

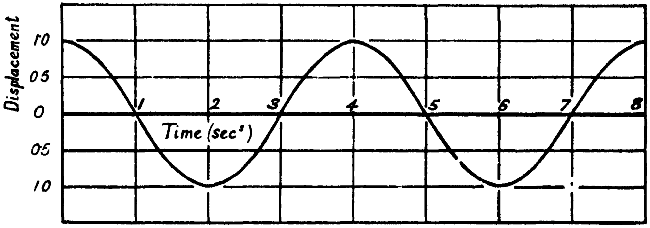


FIG. 1*a*.

Here $n = 2\pi/T = \frac{1}{2}\pi$; hence the displacement at time t is

$$x = \cos \tfrac{1}{2}\pi t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8a)$$

(ii) The effect upon (i) of introducing a damping factor $f = 0{\cdot}2$. (This may be regarded as "strong" damping) (fig. 2*a*).

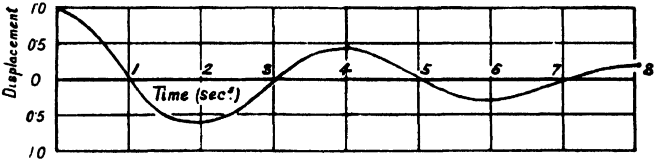


FIG. 2*a*.

Here $f^2/n^2 = 4/25\pi^2 = 0{\cdot}0162$

and $1 - f^2/n^2 = 0{\cdot}9838$

so $$q = \tfrac{1}{2}\pi(0{\cdot}9838)^{\frac{1}{2}} = \tfrac{1}{2}\pi \times 0{\cdot}9918$$

or $$q = \tfrac{1}{2}\pi/1{\cdot}008.$$

Thus the period is lengthened by 0·8 per cent.

The equation for the displacement x' at time t is then

$$x' = e^{-0{\cdot}2t}\cos(\tfrac{1}{2}\pi t/1{\cdot}008) \quad \ldots\ldots\ldots\ldots\ldots\ldots(9a)$$

(It is better to leave the expression inside the brackets in the above form, in order to effect an easy comparison with (8a).)

A number of corresponding values of x, x' and t are given in the table on p. 205, while the graphs plotted from them are given in figs. 1a, 2a.

Exercises on Note 2a.

1. What is the nature of the motion determined by equation (1a) when $f = n$?

2. Calculate the value of the logarithmic decrement for the example given in Note 2a.

   (Ans. 0·4032.)

3. In one of Kater's experiments (see Experiment XXXVIII.) with a seconds-pendulum, the amplitude of the oscillation was decreased in 250 complete periods in the ratio 1 : 1 . 95, by air-friction.

   Calculate the logarithmic decrement; and hence determine the damping factor f, and the percentage increase in the period due to damping.

   (Ans. log. dec. = 0·000356
   f = 0·000356
   increase in period = $6{\cdot}44 \times 10^{-7}$ per cent.)

4. The student would find it instructive to measure the logarithmic decrement for the oscillation of a wave-tracing spring such as used in Experiments XXI. ff. The necessary measurements may easily be obtained from the wave-tracing itself.

Note 3. Motion of a simple pendulum.

To show that a simple pendulum performs Simple Harmonic Motion about the lowest point O of its path, provided that the angle through which it oscillates on either side of the vertical is sufficiently small for its sine

(*continued on p 206*)

Undamped Oscillation $x = \cos \frac{1}{2}\pi t$.		Damped Oscillation $x' = e^{-0\cdot 2t} \cos (\frac{1}{2}\pi t/1\cdot 008)$.	
t.	x.	t.	x'.
0·0	**1·000**	**0·000**	**1·000** $=$ **1·000**
0·5	0·707	0·504	$e^{-0\cdot 1008} \cos \frac{\pi}{4} = 0\cdot 904 \times 0\cdot 707 = 0\cdot 642$
1·0	0·000	1·008	0·000 $=$ 0·000
1·5	−0·707	1·512	$-e^{-0\cdot 3024} \cos \frac{\pi}{4} = -0\cdot 739 \times 0\cdot 707 = -0\cdot 522$
2·0	**−1·000**	**2·016**	$-e^{-0\cdot 4032}$ $=$ **−0·668**
2·5	−0·707	2·520	$-e^{-0\cdot 5040} \cos \frac{\pi}{4} = -0\cdot 604 \times 0\cdot 707 = -0\cdot 427$
3·0	0·000	3·024	0·000 $=$ 0·000
3·5	0·707	3·528	$e^{-0\cdot 7056} \cos \frac{\pi}{4} = 0\cdot 494 \times 0\cdot 707 = 0\cdot 349$
4·0	**1·000**	**4·032**	$e^{-0\cdot 8064}$ $=$ **0·446**
4·5	0·707	4·536	$e^{-0\cdot 9072} \cos \frac{\pi}{4} = 0\cdot 403 \times 0\cdot 707 = 0\cdot 285$
5·0	0·000	5·040	0·000 $=$ 0·000
5·5	−0·707	5·544	$-e^{-1\cdot 1088} \cos \frac{\pi}{4} = -0\cdot 330 \times 0\cdot 707 = -0\cdot 233$
6·0	**−1·000**	**6·048**	$-e^{-1\cdot 2096}$ $=$ **−0·298**
6·5	−0·707	6·552	$-e^{-1\cdot 3104} \cos \frac{\pi}{4} = -0\cdot 270 \times 0\cdot 707 = -0\cdot 191$
7·0	0·000	7·056	0·000 $=$ 0·000
7·5	0·707	7·560	$e^{-1\cdot 5120} \cos \frac{\pi}{4} = 0\cdot 220 \times 0\cdot 707 = 0\cdot 156$
8·0	**1·000**	**8·064**	$e^{-1\cdot 6128}$ $=$ **0·199**
		This is a strongly-damped oscillation; but even here it will be noticed that, while the amplitude is reduced to one-fifth of its initial value after two complete oscillations, the period is only increased by 0·8 per cent.	

to be sensibly equal to its radian measure ;* and that, under these conditions, the time of oscillation T is given by

$$T = 2\pi\sqrt{L/g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where L is the length of the pendulum.

Let AP (fig. 2) be a simple pendulum; that is, a particle P of mass m suspended by a weightless inextensible string AP of length L.

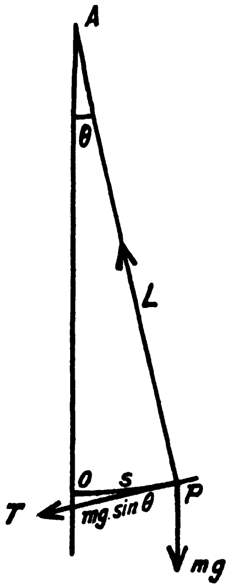


Fig. 2.

Let the pendulum be set in motion by swinging it up out of the vertical, and then releasing it.

At a given instant of the subsequent motion, let

$$\text{angle OAP} = \theta,$$
$$\text{arc OP} = s.$$

Now consider the forces acting upon P at this instant; these are

(1) The tension in the string.
This acts along PA, and so has no resolved part in the direction of P's motion (i.e. along PT).

(2) The weight mg of the particle P.
This has a resolved part

$$mg \sin \theta$$

in the direction PT.

Thus the acceleration $\ddot{s}$ of P along the tangent to its path is

$$\begin{aligned}\ddot{s} &= -mg \sin \theta/m = -g \sin \theta \\ &= -g(\theta - \theta^3/3!\ldots + \ldots) \\ &= -g\theta,\end{aligned}$$

---

* Reference to Tables will show that the two are equal to 4 places of decimals for angles up to 4°.

if θ be sufficiently small for θ^3 to be neglected in comparison with it.

But $\theta = s/L$,

hence $\ddot{s} = -gs/L$,

so that the motion is Simple Harmonic, and of period

$$T = 2\pi/n, \quad \text{where } n^2 = g/L,$$

that is $$T = 2\pi\sqrt{L/g} \quad \ldots\ldots(5)$$

The following table, taken from Wagstaff's *Properties of Matter*, shows how the period departs from that given by (5) when the angle through which the pendulum swings is no longer small:—

θ.	T.
0°	1
2°	1·00008
4°	1·0005
10°	1·002
20°	1·008
30°	1·02
60°	1·07
90°	1·18

For further information on Harmonic Motion from the experimental point of view, the student is referred to Searle's *Experimental Harmonic Motion* (Camb. Univ. Press).

Note 4. The oscillation of a spiral spring.

To show that a spiral spring, oscillating vertically under a load which gives a steady extension L,* performs S.H.M. of period

$$T = 2\pi\sqrt{L/g},$$

i.e. it is isochronous with a simple pendulum of length L.

---

* Provided that it is not extended beyond its elastic limit. (See Note 1, above.)

It is also to be observed that all the conclusions of this Note apply equally to an elastic string, since the law of extension is the same.

Let the spring (fig. 3) be suspended vertically from the point A, and let a load be attached to the lower end B which gives it a steady extension BC ($= L$).

To start the vertical oscillation, pull the spring down below C, and then release it.

Now consider the forces acting on the spring at a subsequent instant of the motion when the lower end is at the point P. Let CP $= x$.

These forces are

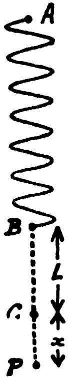


FIG. 3.

(1) A force F' due to the load. When the spring is at rest under its action, an extension L is produced by it.

If a be the unstretched length of the spring we have

$$F' = kL/a.^*$$

This force acts vertically downwards at P.

(2) A restorative force F'' due to the tendency of the spring to regain its unstretched length.

This force acts vertically upwards, and must be equal to that which would give a steady total extension $L + x$, i.e.

$$F'' = k(L + x)/a.$$

Hence the resultant upward force is $F'' - F'$, i.e., it is

$$kx/a,$$

so that the acceleration (upwards) at this instant is

$$kx/am,$$

where m is the mass with which the spring is loaded (the mass of the spring itself is here neglected).

Thus the acceleration is proportional to the displacement from the position C of steady extension, and is directed towards this point, i.e. the motion is S.H., and of period

$$T = 2\pi(am/k)^{\frac{1}{2}} \qquad \text{(6)}$$

Now $F' = mg = kL/a,$

---

* Where k is the elastic modulus of the spring (*ibid.*).

thus $$am/k = L/g \qquad (7)$$
Substitution of (7) in (6) gives

$$T = 2\pi\sqrt{L/g},$$

which proves the proposition.

Exercises on Note 4.

(In order to reduce the Arithmetic, in these examples take g as 32 in ft.-sec. units, and as 981 in cm.-sec. units.)

1. What are the dimensions of the elastic modulus k in terms of the fundamental units of mass, length and time?

   Give a verbal definition of k without the use of any formula.

2. A spiral spring of natural length 1 foot is suspended vertically by one end, while to the other end are attached successively weights of 16, 4, and 1 oz. If the elastic modulus of the spring is 2 lbs. wt., find the steady extension produced in each case.

   (Ans. 6 in., 1·5 in., 0·375 in.)

3. Find the period of a small vertical oscillation in the three cases of Ex. 2.

   (Ans. $\pi/4$, $\pi/8$, $\pi/16$,
   or 0·785, 0·393, 0·197 sec.)

4. A mass of 10·9 gms. is suspended from an elastic string of natural length 36 cm. The periodic time of a small vertical oscillation is found to be 0·785 second. Find the elastic modulus of the string.

   (Ans. 25·6 gms. wt.)

5. Find the steady extension produced in the previous example.

   (Ans. 15·3 cm.)

6. A uniform rod AB of weight 1·6 lbs. rests with the end B upon a smooth horizontal plane, while the end A is suspended by an elastic string of natural length 1 foot, whose elastic modulus is 80 poundals.

   Find the extension produced in the string.

   (Ans. 0·32 ft.)

7. A metal block rests on a horizontal table (coeff. of friction $= \mu$). It is attached to a nail in the table by a (horizontal) piece of elastic whose weight is negligible. The

elastic is stretched to twice its natural length, and the block is then released. Prove that

(*a*) If the elastic modulus be 2μ times the weight of the block, the latter will come (permanently) to rest when the string has contracted to its natural length;

(*b*) If the elastic modulus be 4μ times the weight of the block, this latter will just reach the nail.

Note 5. To show that (if the angular displacement be small) the Mean-position-velocity of a simple pendulum is directly proportional to the amplitude of the arc of swing.

It has been shown in **Note 3** above that, if the angular displacement is small, the motion of a simple pendulum is Simple Harmonic.

If s denote the displacement along the arc at time t, measured from the lowest, or mean, position, we have, following the notation of **Note 3**,

$$s = a \,.\, \cos nt,$$

where $n = g/L$.

Differentiating with regard to t, we have for the velocity along the arc

$$\dot{s} = -an \,.\, \sin nt.$$

The mean-position-velocity v_m is the value of $\dot{s}$ when t is a quarter period, i.e. when

$$nt = \tfrac{1}{2}\pi.$$

thus $\quad v_m = -ag/L.,$

so that $\quad v_m/a = \text{constant},$

which proves the proposition.

Cor.—It is to be observed that the constant in the last equation does not involve the mass of the pendulum; hence

the mean-position-velocity is independent of the mass of the pendulum.

Note 6. To show that the radius of gyration k, about its axis, of a solid cylindrical wheel of radius a is given by

$$k^2 = \tfrac{1}{2}a^2.$$

The actual wheel (mass M), and the imaginary substitute (also of mass M) consisting only of a rim of radius k, must have the same kinetic energy when rotating with angular velocity ω.

Let σ be the mass per unit area of cross-section. If the wheel be of uniform thickness, this thickness does not enter into the calculation. (It is here also presumed that the wheel is of uniform density.)

To calculate the total kinetic energy of the wheel, we imagine it to be built up of an infinite number of infinitesimally thin rings concentric with the axle.

Consider such an elementary ring between radii r and $r + dr$ (fig. 4).

The cross-sectional area of this ring is

$$2\pi r \, . \, dr$$

so that its mass is

$$2\pi r\sigma \, . \, dr,$$

and its peripheral * velocity is

$$r\omega,$$

hence its kinetic energy dE is

$$dE = \tfrac{1}{2} \, . \, 2\pi r\sigma r^2\omega^2 \, . \, dr.$$

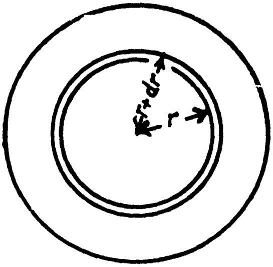


FIG. 4.

Summing for all the elementary rings composing the wheel, we have that the total kinetic energy E is

$$E = \pi\sigma\omega^2\int_0^a r^3 dr = \tfrac{1}{4}\pi\sigma\omega^2 a^4$$
$$= \tfrac{1}{4}Ma^2\omega^2, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

since $\pi a^2\sigma = M$.

Now the peripheral velocity of the equivalent " rim-wheel " is

$$k\omega.$$

so that its kinetic energy is

$$\tfrac{1}{2}Mk^2\omega^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

equating (8) and (9) we have

$$\tfrac{1}{2}Mk^2\omega^2 = \tfrac{1}{4}Ma^2\omega^2,$$

$$\text{i.e. } k^2 = \tfrac{1}{2}a^2.$$

Exercise on Note 6.

Prove that the radius of gyration k of a sphere of radius a about a diameter is given by

$$k^2 = 2a^2/5.$$

---

* That is, the velocity along the periphery or circumference.

(The sphere may be regarded as built up of an indefinite number of indefinitely narrow discs whose centres lie along the diameter in question; the radius y of any disc being connected with the distance x of its centre from the middle point of the diameter by the relation

$$x^2 + y^2 = a^2.)$$

Note 7. To prove that, for a rectangular rod of length $2a$ and depth $2b$,* the radius of gyration K about the axis through the centroid perpendicular to its length and depth is given by

$$K^2 = (a^2 + b^2)/3.$$

We are required to find two points, one on each side of the axis such that, if a mass equal to half that of the rod were concentrated at each of these points, then

Total Kinetic Energy of the whole rod when revolving—say with unit angular velocity—about the given axis

= Kinetic Energy of these two particle-substitutes when revolving with the same angular velocity about the same axis.

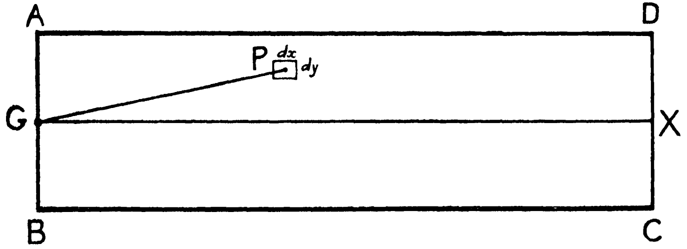


Fig. 5.

From considerations of symmetry, the two points will be equidistant from the axis; we need, therefore, only do the calculation for half the rod.

Let ABCD (fig. 5) be a section of the right half of the rod by a plane perpendicular to the axis through the centroid G.

---

* If the rod be of uniform width parallel to the axis, this width does not enter into the calculation.

Since all sections in this direction are the same shape, the width parallel to the axis need not enter into the calculation.

Take the lines GX, GA as co-ordinate axes, and let M be the mass of this half-rod, while σ is the mass per unit area of the cross-section parallel to the plane of the paper.

Consider the elementary area $dx \, . \, dy$ surrounding the point P (x, y).

Its mass is $\sigma \, . \, dxdy$,

and its Kinetic Energy when revolving with unit angular velocity about G is

$$\begin{aligned} dE &= \tfrac{1}{2}\sigma \, . \, dxdy \, . \, \mathrm{GP}^2, \\ &= \tfrac{1}{2}\sigma(x^2 + y^2) \, . \, dxdy. \end{aligned}$$

Summing this for all the elementary areas composing the figure, we have that the total Kinetic Energy E is

$$E = \iint \tfrac{1}{2}\sigma(x^2 + y^2) \, . \, dxdy,$$

where the limits of integration remain to be assigned.

The limits of x are 0 and a, while those of y are $-\, b$ and $+\, b$; further, the limits of y are the same for all values of x, hence the x- and y-integrations may be performed independently.

$$\text{Thus} \quad E = \tfrac{1}{2}\sigma\left(\int_0^a x^2 dx \, . \int_{-b}^{b} dy + \int_0^a dx \, . \int_{-b}^{b} y^2 dy\right)$$

$$= \sigma(a^3 b + ab^3)/3.,$$

$$\text{i.e.} \qquad E = M(a^2 + b^2)/6, \qquad \text{since } 2ab\sigma = M.$$

Now the Kinetic Energy E' of the particle-substitute, if K be its distance from G, is

$$E' = \tfrac{1}{2}M \, . \, K^2.$$

Equating E and E', we have

$$\tfrac{1}{2}M \, . \, K^2 = M(a^2 + b^2)/6$$

$$\text{so that} \qquad K^2 = (a^2 + b^2)/3.$$

The Moment of Inertia of the rod about this axis is obtained, if it be required, by multiplying the square of K by the mass of the rod.

Note 8. Angular motion resulting from an impulsive blow.

Problem : A rigid body, at rest and free to rotate about an axis, receives a blow (impulse) in a plane perpendicular to the axis. It is required to find an expression for the angular velocity ω *with which the body begins to rotate about the axis.*

Let P (fig. 6) be any particle, mass m, of the body ; which is at rest, but free to rotate about an axis through O at right angles to the plane of the paper. Let OP = r.

Now suppose that P receives an impulse, in the plane of the paper, whose components at right angles to, and along OP are $X + X'$, $Y + Y'$, respectively; where X, Y are due to the external action, while X', Y' are due to reactions with neighbouring particles.

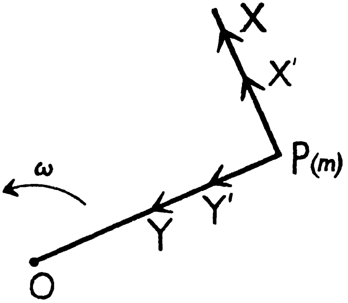


Fig. 6.

If v be the velocity with which P begins to move, we have

$$X + X' = mv.$$

Multiplying by r,

$$(X + X')r = mvr = mr^2\omega.$$

Summing this for all the particles of the body

$$\Sigma(X + X')r = (\Sigma mr^2)\omega = I\omega \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

where I is the moment of inertia of the body about the axis.

Since Y' passes through O, $\Sigma X'r$ represents the sum of the moments of the internal impulses about O, and this by Newton's Third Law is zero.

Hence (10) becomes

$$\Sigma Xr = I\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

which is the required equation for ω.

Note 9. Theory of the motion of the Impact-arm in Experiment XXVIII.

We have to show that the motion of the Impact-arm and its load immediately after the collision is equivalent to the motion along the line of impact of a certain mass J (the "equivalent trolley"); and it is required to find the value of J in terms of the constants of the Impact-arm and its load.

In order to calculate the angular velocity ω of the Impact-arm immediately after the collision, we make use of equation (11) of Note 8.

If c be the distance of the line of impact from the axis of rotation of the arm, then, using the notation of Experiment XXVIII, equation (11) gives

$$I\omega = Pc = cW(U_1 - U_2)\ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

where I is the moment of inertia of the Impact-arm and its load about the axis of rotation, and P is the impulse of the blow given by the bombarding trolley.

In order to express this result in terms of V, the velocity actually recorded by the wave-tracing device on the platform of the impact-arm, we notice, on referring to the experiment, that c is also the distance of the zero-line of this wave-tracing from the axis, so that

$$V = c\omega.$$

Equation (12) then gives

$$W(U_1 - U_2) = (I/c^2).V\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

Thus the motion of the Impact-arm immediately after the collision may be regarded as equivalent to that of a trolley of mass

$$J = I/c^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

moving along the line of impact.

Exercise on Note 9.

Regarding the Impact-arm as a rectangular beam (mass M) of length $2a$ and breadth $2b$, and the load as 6 equal cylinders (each of mass m, and radius r), which are arranged in pairs with their axes parallel to the axis of rotation, and separated from it by distances d_1, d_2, d_3, respectively: obtain an expression for J in terms of these constants, and make use of it to check the numerical value of J obtained when you perform Experiment XXVIII.

Note 10. The motion of a compound pendulum.

Consider a body of mass M gms. oscillating about a horizontal axis through the point O.

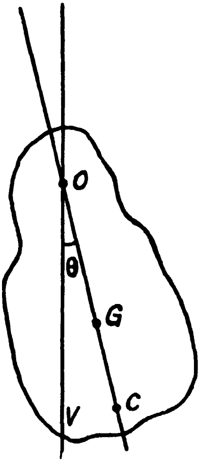


FIG. 7.

Let G be the centroid of the body. Fig. 7 shows a vertical section through G. In the equilibrium position the line OG will evidently be vertical (the line OV in the figure).

The oscillation is started by turning the body about the axis through O until OG makes an angle α with the vertical. The body is then released.

Let OG = h cm.; then when the angle VOG has decreased to θ—say, after t seconds—the centroid has fallen a distance

$$h(\cos\theta - \cos\alpha) \qquad \text{cm.}$$

Hence the work done on the body by gravity is

$$Mgh(\cos\theta - \cos\alpha) \qquad \text{ergs.}$$

Let k be the radius of gyration about the horizontal axis through O; then we may, for dynamical purposes, replace the actual body by a single particle of mass M whose distance from O is k. This particle will have the same angular velocity $\dot{\theta}$ as the actual body.

We thus see that the total Kinetic Energy of the body at this instant is

$$\tfrac{1}{2}Mk^2\dot{\theta}^2 \text{ ergs.}$$

This expression also represents the change of Kinetic Energy from the position α to the position θ, since in the former position the Kinetic Energy is zero.

Employing the principle of Experiment XXVI., viz.:—

Change of Kinetic Energy = Work done,

we have $$\tfrac{1}{2}Mk^2\dot{\theta}^2 = Mgh(\cos\theta - \cos\alpha).$$

Differentiating with regard to t

$$Mk^2\dot{\theta} \,.\, \ddot{\theta} = -\,Mgh \,.\, \sin\theta \,.\, \dot{\theta},$$

or, dividing by $Mk^2\dot{\theta}$,

$$\begin{aligned}\ddot{\theta} &= -\,(gh/k^2) \,.\, \sin\theta \\ &= -\,n^2\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)\end{aligned}$$

if θ, supposed measured in radians, be small, and n^2 stands for gh/k^2.

Thus the motion is Simple Harmonic, and its period T is given by

$$T = 2\pi/n,$$

i e. $$T = 2\pi\sqrt{k^2/hg} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

Comparing equation (16) with equation (5), Note 3, we see that the period **is the same as that of a Simple pendulum of length**

$$k^2/h.$$

For a body of irregular shape the accurate determination of h and k is a matter of some difficulty; the problem is, however, greatly simplified by the fact that there exists a point C on the line OG such that **the period of oscillation of the body about C is the same as that about O.**

This point C is called the **Centre of Oscillation,** while O is called the **Centre of Suspension.**

Before establishing the existence of the point C, it is necessary first to express the period T of the body about O in terms of the radius of gyration K about the axis through the centroid G parallel to the axis through O.

Now, it is a well-known theorem * that

The square of the radius of gyration of a body about any given axis = the square of the distance of this axis from the centroid, together with the square of the radius of gyration about a parallel axis through the centroid;

i.e. $$k^2 = h^2 + K^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

Substitution from (17) in (16) gives

$$T = 2\pi\sqrt{(h^2 + K^2)/hg} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

---

* See, for example, Fawdry's *Dynamics*, Part II. (G. Bell & Co.).

To prove that the Centres of Oscillation and Suspension are convertible.

Let C be the point in OG such that the length OC ($= L$) is that of a Simple pendulum whose period is the T of equations (16) and (18),

i.e.
$$L = (h^2 + K^2)/h$$
$$= h + K^2/h \text{ ....(19)}$$

Suppose now that the body be inverted, and made to oscillate about an axis through C parallel to the original axis through O.

The period T' of this oscillation will evidently be given by

$$T' = 2\pi\sqrt{(h'^2 + K^2)/h'g} \text{ ....(20)}$$

where $h' =$ CG.

But from (19), $h' = L - h = K^2/h$, so that

$$\frac{h'^2 + K^2}{h'g} = \frac{K^4/h^2 + K^2}{gK^2/h} = \frac{h^2 + K^2}{hg},$$

thus $T' = T$,

i.e. the periods of oscillation about the points O and C are equal.

Kater's Pendulum (see Experiment XXXVIII.) employs the principle of the convertibility of the centres of oscillation and suspension.

The pendulum consists of a graduated rod carrying two parallel, but opposed, knife-edges whose positions along the rod are adjustable. The rod also carries heavy sliding weights which may be clamped in any position.

Having clamped the weights in a suitable position, positions of the knife-edges are found by trial such that the period of oscillation of the pendulum about either knife-edge is the same; it then follows from the above investigation that the pendulum is equivalent to a Simple pendulum whose length is equal to the distance between the knife-edges.

Exercises.

1. If the pendulum of fig. 7, when hanging at rest with the line OC vertical, be struck a horizontal blow whose impulse is P (dyne-secs.) at a point distant x cm. vertically below O, prove that there is an impulsive reaction R at O, whose magnitude is given by

$$R/P. = 1 - \qquad /(h^2 + K^2).$$

2. From the result of Exercise 1, prove that there is no reaction at O when the line of action of the blow passes through the Centre of Oscillation C.*

3. With the help of equation (18), prove that the period T of a quasi-simple pendulum consisting of a heavy sphere of radius a, suspended by a string of length $L - a$ is, neglecting the weight of the string,

$$T = 2\pi\{(L/g)(1 + 2a^2/5L^2)\}^{\frac{1}{2}}.$$

Hence show that, if a be small compared to L; and if T_0 be the period of a true simple pendulum of length L, then

$$T : T_0 :: (1 + a^2/5L^2) : 1 \text{ (approximately).}$$

(The expression for the radius of gyration of a sphere about a diameter is given in Appendix I., § 19. See also Appendix II., Note 6, Exercise.)

4. Referring to Exercise 3, calculate the percentage increase of T over T_0 for the cases where
   (i) $a = 2$ cm., $L = 100$ cm.,
   (ii) $a = 10$ cm., $L = 50$ cm.,
   (iii) $a = 0{\cdot}625$ in., $L = 45{\cdot}38$ in. (The "simple pendulum" employed in Experiment XXXVIII.)

(Ans. (i) 0·008 per cent.
(ii) 0·8 per cent.
(iii) 0·0038 per cent.)

---

* For this reason the Centre of Oscillation has also been called the **Centre of Percussion.** This principle has an application in the cricket bat. The bat should be so constructed and so employed, that the point at which the ball hits it is the Centre of Percussion corresponding to the point in the handle about which the bat is swung; an attempt to take the ball with a different part of the bat will lead to an unpleasant jarring of the wrists.

A very simple and ingenious experiment illustrating the properties of the Centre of Percussion is given in Clerk Maxwell's *Matter and Motion* (S.P.C.K.), which the student is recommended to study.

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